The multiple hierarchical legislatures in representative democracy *

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Abstract

Multiple hierarchical models of representative democracies in which, for instance, voters elect county representatives, county representatives elect district representatives, district representatives elect state representatives and state representatives elect a president, reduces the number of electors a representative is answerable for, and therefore, considering each level separately, these models could come closer to direct democracy. In this paper we show that worst case policy bias increases with the number of hierarchical levels. This also means that the opportunities of a gerrymanderer increase in the number of hierarchical levels.

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1 Introduction

In many parliamentary democracies each representative is elected within a single-member district. However, in general, since representatives are very fewer than voters, there are too many voters for the representative who is the prime minister elected in the parliament to answer voters' demands for policies in a district. As a result, this might lead to a misrepresentation of voters' interests, lower participation rates and other negative effects. A possible extension of this model could reduce the misrepresentation and the other negatives with *intermediate districts* and representatives. Voters in the same single-member district would elect an intermediate representative, who grouped together with an appropriate number of other

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intermediate representatives into an intermediate single-member district could vote for one of the final representatives. Clearly, one can extend this model further by increasing the number of intermediate levels, and thus, obtaining a model of multiple hierarchical representative democracy.

In fact our suggested model can be basically found in the literature and they were, though not in its pure form, attempts for its implementation in the past one and a half centuries. In expressing his views on democracy, Jefferson (1816) outlined a so-called ward system in which he distinguished between the national, the state, the county and the ward level. He characterized this system by

> "It is by dividing and subdividing these republics from the great national one down through all its subordination, until it ends in the administration of every man's farm by himself; by placing under every one what his own eye may superintend, that all will be done for the best."

The so-called council system is based on a similar idea (for its history, we refer to Olsen, 1997). However, the council system only existed just formally, in the sense that contrary to its spirit it did not result in a democracy, but merely in a bureaucratic executing system of a totalitarian state. Though our real-life experiences with council systems are extremely negative, it still has its advocates like Arendt (1977). [1] From another point of view understanding council systems may help us in understanding democracy as stated by Medearis (2004):

> "An understanding of the council movement's ideas and practices might have (and could still) contribute to an enrichment of democratic theory. ... The majority of movement participants did not reject parliamentary institutions, but attempted to achieve their democratizing goals in tandem with them, apparently envisioning an interplay between different social entities that would bring about the desired unleashing of democratic agency."

We are not aware of a formal analysis proving why the council system might have gone wrong. In this paper, by extending the model of districting by Gilligan and Matsusaka (2006), we show that in the worst-case or in the case of extreme gerrymandering a multiple hierarchical model of representative democracy can serve the interest of a minority. In particular, by increasing the number of intermediate levels we get closer and closer to a dictatorship. Concerning random districting the same results as obtained by Gilligan and Matsuasaka (2006) prevail for the multi-level hierarchical model of representative democracy. In particular, policy bias emerges with positive probability and expected bias equals zero in a symmetric setting, while in a skewed setting expected bias is also positive.

The remainder of the paper is organized as follows. Section 2 presents our extended model of multiple hierarchical representation and a motivating example in the case of extreme district

[1] For a discussion of Arendt's work see, for instance, Isaac (1994).

maker. Section 3 considers the worst case bias, which can emerge by coincidence or in the case of an optimal gerrymandering. Section 4 investigates random districting. Section 5 shows the case of a moderate district maker. Finally, we conclude in Section 6.

2 The model

We use a multiple application of the median voter theorem by extending Gilligan & Matsusaka's model that is a double application of it. According to the median voter model introduced by Black (1958) which is well known, the policy which the median voter prefers prevails over any other policies in the case of a uni-dimensional policy space and single-peaked preferences of voters.

Gilligan & Matsusaka define and calculate a bias between the median policy and the policy decided in the legislature composed of representatives elected in single-member districts, where each voter is allocated to exactly one district. They mention that there is a possibility for gerrymandering in the indirect democracy that is to divide voters into districts for giving one group an advantage in the election, and that as a result, the final policy chosen by representatives in the legislature may be away from the policy the median voter prefers.

As we pointed out in the introduction, the political system in which the policy is decided hierarchically is adopted in some societies. In this paper, by extending Gilligan & Matsusaka's model, more accurately extending the median voter theorem vertically, we will show in the following part how the final policy, which the representative being finally elected in "multiple levels legislatures" chooses and implements, is decided in gerrymandered districts and we will show how the final policy can deviate from that of the median voter theorem.

Basically, the settings and notations of our model are similar to Gilligan & Matsusaka's model. The population of citizens consists of N assumed as an odd number of people, all of them vote, and they are called voters hereafter, so that the voters set is defined as $\mathcal{N} = \{1, 2, 3, \ldots, N\}$. We assume that each voter $i \in \mathcal{N}$ has an ideal policy at own position $x_i \in \mathbb{R}$, and that each voter's utility decreases strictly monotonically as an implemented policy is getting farther away from its own ideal position. Then the median of all voters is voter $\frac{N+1}{2}$. We also assume that voters with smaller numbers (the left wing from the median) are more liberal and that those with larger numbers (the right wing from the median) are more conservative for convenience. Thus we label the voters such that $x_1 \leq x_2 \leq \ldots \leq x_N$. Let $F(x)$ be the cumulative distribution function of voter ideal points. We assume there is a unique median voter in the population with ideal point x^*_{POP} such that $F(x^*_{POP}) = \frac{1}{2}$. The distance of a policy x from the median of all voters is called a bias, which we define below in an analogous way to Gilligan & Matsusaka:

Definition 1. *The measure of a policy bias equals* $B = |F(x) - F(x^*_{POP})|$.

Observe that policy choice x is unbiased or has minimal bias when $B = 0$, and x has maximal bias when $B = 1/2$.

We consider $t+1$ decision levels starting from $t = 0$. On each decision level $i \in \{1, 2, \ldots, t, t+1\}$ voters grouped into equally sized districts send a representative to the next, the $i+1$-th decision level. On the t-th decision level, representatives are elected and sent to the $t+1$-th decision level, which is the final decision level. On the $t+1$-th decision level, where only one district is formed, only one representative who decides the final policy is elected. We assume that none of the representatives can make a binding commitment regarding the final policy, so that the final representative chooses and implements its most favored policy, that is its own ideal policy.

Since now we are considering hierarchical legislatures composed of representatives elected among voters, the above decision process can also be described in the following way: the lowest decision level consists of the voters, the second decision level is legislature 1, the third decision level is legislature 2, and so on. Finally the t-th decision level is legislature $t-1$, and the $t+1$-th decision level is legislature t that is composed of only one district and that elects only one representative who is called the final representative. Thus t means the number of hierarchical legislatures inserted between voters and the final representative, which are like ward representatives, county representatives, state representatives and the national representatives.

Each legislature is composed of representatives as follows. Let K_i be the number of districts on the i-th decision level, $i \in \{0, 1, 2, \ldots, t, t+1\}$. Since on the $t+1$-th decision level there is only one district, we let $K_{t+1} = 1$. We assume single-member districts, consequently K_i is also the number of representatives in legislature i on the $i+1$-th decision level. For convenience, let $K_0 = N$. When $t = 0$, since there are no legislatures between voters and the final representative, only one representative is elected directly among the voters, who implements the final policy, that is viewed as the direct democracy. When $t \geq 1$, our model becomes an indirect democratic system, especially when $t \geq 2$, it has hierarchical multiple decision levels, namely hierarchical legislatures with multi-levels. We assume that these representatives on each decision level are elected among voters or representatives in districts of one level below by a majority rule. We also assume that every voter and representative on each i-th decision level cast a ballot sincerely and that only one among them is elected as a single-member district representative. Consequently the median voter of each district is elected as the representative. Thus the legislature at the $i+1$-th decision level is composed of the median voters from each district from a level below. Since every district at the same level is assumed to have the same size, the population of each district at the $i+1$-th decision level equals K_i/K_{i+1}. Since usually the numbers of districts in upper legislatures are equal or smaller than those in lower legislatures, $K_{i+1} \leq K_i$ is assumed. Unless anything is mentioned, $K_{i+1} < K_i$ is assumed.

The basic structure of our model is like the following. First, N voters divided into K_1 equal-sized districts with $N/K_1 = K_0/K_1$ voters per district at the first decision level elect the representatives of legislature 1 at the second decision level, consequently K_1 representatives are in legislature 1. Second, the representatives of legislature 1 divided into K_2 equal-sized districts with K_1/K_2 members per district elect the representatives of legislature 2 at the third decision level, consequently K_2 representatives are in legislature 2. Third, the representatives of legislature 2 divided into K_3 equal-seized districts with K_2/K_3 members per district elect the representatives of legislature 3 at the forth decision level, consequently K_3 representatives are in legislature 3, and so on. Finally, since the $t+1$-th decision level is the final one and $K_{t+1} = 1$, the K_t representatives in legislature t at the $t+1$-th decision level elect only one representative who is the final representative. The representative decides and implements only one policy, which is a number on the real line, applying to all voters. Table 1 depicts the above structure. Here, since all voters and representatives cannot commit to policies at all, the final representative implements its own ideal policy. We can say that Gilligan & Matsusaka's model is the special case of $t = 1$ in our model. Additionally, since every district is of equal-size at each level, there is no vote-value disparity.

Table 1: The basic structure of the model

decision level	voter or legislature	number of district		total population	population per district
first	voters	K_1		$K_0 = N$	N/K_1
			$\searrow$		
second	legislature 1	K_2		K_1	K_1/K_2
			$\searrow$		
third	legislature 2	K_3		K_2	K_2/K_3
.	.	.	$\searrow$	.	.
.	.	.		.	.
.	.	.	$\searrow$	.	.
$t-1$-th	legislature $t-2$	K_{t-1}		K_{t-2}	K_{t-2}/K_{t-1}
			$\searrow$		
t-th	legislature $t-1$	K_t		K_{t-1}	K_{t-1}/K_t
			$\searrow$		
$t+1$-th	legislature t	$K_{t+1} = 1$		K_t	$K_t/K_{t+1} = K_t$
	$\parallel$				
	t legislatures		$\searrow$		
	final representative			1	

From the above structure, we can immediately obtain the lemma below:

Lemma 1. *The number of hierarchical levels is at most the number of prime factors of N.*

Proof. We will prove this by contradiction. Let $N = a_1 \cdot a_2 \cdot a_3 \ldots a_t \cdot a_{t+1}$, where each a_i, $i \in \{1, 2, 3, \ldots, t, t+1\}$, is a prime factor of N. We assume that for given N we can make $j+1$ decision levels, noting $K_{j+1} = 1$, where $j > t$. Then the populations of each district at each level $1, 2, 3, \ldots, j, j+1$ are $\frac{N}{K_1}, \frac{K_1}{K_2}, \frac{K_2}{K_3}, \ldots, \frac{K_{t-1}}{K_t}, \ldots, \frac{K_j}{K_{j+1}}$, respectively, where $K_{j+1} = 1$. Thus

$$N = \frac{N}{K_1} \cdot \frac{K_1}{K_2} \cdot \frac{K_2}{K_3} \cdots \frac{K_{t-1}}{K_t} \cdots \frac{K_j}{K_{j+1}},$$

and the number of the factors of N is $j+1$. However, N cannot be the product of more than t integers larger than 1 from its prime factorization; a contradiction. $\square$

Hereafter, for convenience, $t+1$ is regarded as the maximum number of decision levels and the number of the prime factors of N. Note that we can set up hierarchical decision levels between 0 and $t+1$ by multiplying any prime factors of N with each other. For instance, supposing $N = \frac{N}{K_1} \cdot \frac{K_1}{K_2} \cdot \frac{K_2}{K_3} \cdots \frac{K_t}{K_{t+1}}$, we can also set up $t-1$ decision levels in addition to $t+1$ levels if we multiply the first three prime factors of $\frac{N}{K_1} \cdot \frac{K_1}{K_2} \cdot \frac{K_2}{K_3}$ with each other and consider the population per district $\frac{N}{K_3}, \frac{K_3}{K_4}, \ldots, \frac{K_t}{K_{t+1}}$ at decision levels, respectively. Also see Example 1 below.

At first sight, it might look like Lemma 1 imposes a severe restriction on the number of possible decision levels in our hierarchical model of representative democracy. For instance, if N is a prime number, we can construct only one district on the first decision level, accordingly $t = 0$ emerges as the only possible case because of the assumption of equal-sized districts. We can overcome this problem by slightly mitigating equal-sized districts to almost equal-sized districts, that is rounding either down or up value K_i/K_{i+1}, and picking consistently one of the two electors from "the middle" in the case of even district sizes, so that the district sizes can be different on the i-th decision level from each other. In this case, some disparities in the number of voters per representation occur on the i-th decision level. However, this extension would just complicate our analysis without substantial gain, and therefore we assume that K_i/K_{i+1} are integers for all $i = 0, 1, \ldots, t-1, t$, where $K_0 = N$, and that districts of all levels have an odd number of voters.

Let $x^*_{j,k}$ be the ideal point of the median representative or voter in district k at the j-th decision level, that is legislature $j-1$. When there are $i+1$ decision levels, at the final decision level where $K_{i+1} = 1$, the policy outcome decided in the final legislature becomes $x^*_{i+1,1}$.

In the next example, we will specifically exhibit how far away the final policy in the multiple hierarchical electoral system in the case of extreme gerrymandering can be from the policy decided in the direct democracy, that is the median voter's ideal policy, and we will also compare the final policy in our model with that in Gilligan & Matsusaka's model in the next example.

Example 1. We will consider an example of three hierarchical legislatures where $t+1 = 3$, $N = 27$, voters set $\mathcal{N} = \{1, 2, 3, \ldots, 25, 26, 27\}$ and voters have different ideal points, that is $x_1 < x_2 < \ldots < x_{27}$. We will compare the indirect democracies of two decision levels ($t = 1$) and three decision level ($t = 2$) with the direct democracy ($t = 0$). [2] In this case, the median

[2] 27 equals $3 \cdot 3 \cdot 3$ by the prime factor decomposition. From this fact, in the case of two decision levels, each district at the first decision level and that at the second level are composed of 9 voters and 3 representatives, respectively, or of 3 and 9, respectively. In the case of three decision levels, each district is composed of three voters or representatives at every decision level. As a result, we can obtain combinations of $K_1 = 1$, $(K_1, K_2) = (3, 1), (9, 1)$ and $(K_1, K_2, K_3) = (9, 3, 1)$ for the one decision level case, in the two decision levels case and the three decision levels case, respectively.

voter is $\frac{N+1}{2} = 14$ as shown in Table 2, so that the final policy is decided at $x^*_{1,1} = x^*_{27} = 14$ in the direct democracy. However, as Gilligan & Matsusaka is showing, the final policy is not

Table 2: The case of $N = 27$, $K_1 = 1$

$$\begin{array}{c} \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27\} \\ \downarrow \\ 14 \end{array}$$

always decided at the median of voters with gerrymandered districts in the indirect democracy. We will exhibit that the selected policy is decided at the father away from the median of voters as the levels of delegation increase in this example. Table 3 and Table 4 illustrate the most extreme gerrymandering to give advantage to the liberal (left) voters in the case of two decision levels ($t = 1$), which can be regarded as regular indirect democracies, where the legislature composed of the representatives elected in single-member districts decides the final policy.

How do liberal voters gerrymander for maximizing their own political payoff? We shall assume that a liberal extremist who is voter 1, hereafter called "she", can arrange all districts without loss of generality. [3] Then she will attempt to put the median of all representatives in each district at each decision level on a position being as left as possible. In this paper, we use the well-known "cracking and packing" algorithm as it is formulated by Gilligan & Matsusaka of which the explanation is in the following quotation for each discrete voter with an ideal point on the real line:

> "first, citizens with high-value ideal points are "cracked" into districts where they are the minority, maximizing the influence of citizens with low-value ideal points, and second, the remaining high value citizens are "packed" into districts containing a preponderance of like-minded citizens in order to waste their votes through overkill."

For $t = 1$ and $N = 27$ we can consider two cases, where $(K_1, K_2) = (3, 1)$ and $(9, 1)$. Then we can easily find the policies that are as close as possible to voter 1 who is the liberal extremist like in the example with nine voters in Gilligan & Matsusaka. In both Tables 3 and 4, the policies are decided by voter 10 whose ideal position is 4 positions away from the median voter. Even if there are several districts patterns in a combination of N and t, it can be shown that the maximum distances between the final policies and the median in the case of an optimal gerrymandering for either the liberal or conservative extremists who are "partisan"s remain identical.

Finally, what is the policy decided by the upper representatives if an additional level of representation is incorporated in the above policy decision process in the case of gerrymandering? In the same way as in our previous two decision levels examples, the districts on the

[3] Of course, in the same way we can also consider a conservative extremist instead of a liberal extremist.

Table 3: The case of $N = 27,\ K_1 = 3, K_2 = 1$

$\{1,2,3,4,5,24,25,26,27\}$	$\{6,7,8,9,10,11,12,13,14\}$	$\{15,16,17,18,19,20,21,22,23\}$
$\downarrow$	$\downarrow$	$\downarrow$
$\{5$	10	$19\}$
	$\downarrow$	
	10	

Table 4: The case of $N = 27,\ K_1 = 9, K_2 = 1$

$\{1,2,27\}$	$\{3,4,26\}$	$\{5,6,25\}$	$\{7,8,24\}$	$\{9,10,11\}$	$\{12,13,14\}$	$\{15,16,17\}$	$\{18,19,20\}$	$\{21,22,23\}$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\{2$	4	6	8	10	13	16	19	$22\}$
				$\downarrow$				
				10				

second decision level are arranged by one liberal extremist in order to enforce the final policy coming as close as possible to her own ideal point. In the case of three decision levels the districts pattern is only $(K_1, K_2, K_3) = (9, 3, 1)$. Table 5 illustrates the case. The policy is decided by voter 8 who is even father away from the median than in the case of two decision levels, that is 6 positions away from the median. As a result, this example is suggesting that

Table 5: The case of $N = 27,\ K_1 = 9, K_2 = 3, K_3 = 1$

$\{1,2,27\}$	$\{3,4,26\}$	$\{5,6,25\}$	$\{7,8,24\}$	$\{9,10,11\}$	$\{12,13,14\}$	$\{15,16,17\}$	$\{18,19,20\}$	$\{21,22,23\}$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
2	4	6	8	10	13	16	19	22
				$\downarrow$				
$\{2$	4	$22\}$	$\{6$	8	$10\}$	$\{13$	16	$19\}$
	$\downarrow$			$\downarrow$			$\downarrow$	
	$\{4$			8			$16\}$	
				$\downarrow$				
				8				

the final policy decided by the representative becomes farther away positions from the median of all voters as hierarchical decision processes are added in the democratic representative system in the case of many voters. Indeed, we will mention this fact generally in Proposition 1 in the next section. Additionally, from the result of Lemma 1, many voters are required to construct a higher hierarchical representative system since we can only construct the decision levels within the number of prime factors of voters number, N at most. Actually, in this example where $N = 27$, the liberal extremist can construct only three decision levels at most, consequently she can slide the final policy to the left side by only 6 position from the median voter's position. Thus this example is also suggesting that an extreme voter needs so many other voters in order to obtain a political advantage by gerrymandering. We will also show this fact generally in Corollary 2 in the next section. □

3 Worst case bias or liberal optimal gerrymandering

From Example 1 we can conjecture that the final policy outcome is getting farther from the median of all voters as we add more hierarchical levels of legislatures, when all districts at each level are arranged by a liberal extremist's gerrymandering. The following lemma determines the policy position finally decided in the case of $i+1$ decision levels, namely the policy position is decided in a political decision system composed of both all voters and $i+1$ hierarchical legislatures including the final representative in the indirect democracy. Note that there are $t+1$ decision levels at most in the case of N voters from Lemma 1 and that we can make decision levels between 0 and $t+1$.

Lemma 2. (Generalization of Gilligan & Matsusaka (2006)) *In the case of liberal gerrymandering the policy outcome decided by the political decision system composed of $i+1$ ($i \in \{0, 1, 2, \dots, t\}$) hierarchical decision levels is*

$$x^*_{i+1,1} = \frac{1}{2}\left(\frac{1}{2}\right)^i \frac{1}{K_i K_{i-1} \cdots K_2 K_1}(K_i+1)(K_i+K_{i-1})(K_{i-1}+K_{i-2}) \cdots (K_2+K_1)(K_1+N). \quad (1)$$

Proof. (1) will be proved by the induction.

(I) Suppose $i = 0$. This case is the direct democracy case with no legislature. When $i = 0$, in the right-hand side of (1), both $\frac{1}{K_i K_{i-1} \cdots K_2 K_1}$ and $(K_i + K_{i-1})(K_{i-1} + K_{i-2}) \cdots (K_2 + K_1)(K_1 + N)$ include no terms. Thus (1) becomes

$$x^*_{1,1} = \frac{1}{2}\left(\frac{1}{2}\right)^0 (K_0 + 1) = \frac{N+1}{2}.$$

This is consistent with the median of all voters.

(II) Suppose $i = 1$. This case is already proved in Proposition 1 of Gilligan & Matsusaka (2006).

(III) Suppose, when $i = n$ ($n \in \{1, 2, 3, \dots, t-1\}$), (1) is correct. Then it is sufficient to show that (1) is correct when $i = n+1$. Noting that $K_{n+1} = 1$, we can transform (1) at $i = n$ to

$$x^*_{n+1,1} = \underbrace{\frac{1}{2}\left(\frac{K_n}{K_{n+1}} + 1\right)}_{\text{median of } n+1\text{-th level}} \underbrace{\frac{1}{2}\left(\frac{K_{n-1}}{K_n} + 1\right)}_{n\text{-th level}} \underbrace{\frac{1}{2}\left(\frac{K_{n-2}}{K_{n-1}} + 1\right)}_{n-1\text{-th level}} \cdots \underbrace{\frac{1}{2}\left(\frac{K_1}{K_2} + 1\right)}_{\text{second level}} \underbrace{\frac{1}{2}\left(\frac{N}{K_1} + 1\right)}_{\text{first level}}.$$

This formula means the following: the last term is the median of the first district at the first decision level, the second term from the last is the median of the first district at the second decision level, and so on. Finally the first term is the term to indicate the $\frac{K_n+1}{2}$-th representative at the $n+1$-th decision level at which there is only one district. By this term, the median in the district on the $n+1$-th decision level is indicated.

Here when $i = n+1$, the final decision level is the $n+2$-th, and K_{n+1} representatives are at the $n+2$-th decision level. Actually, since n is at most $t-1$, when $i = n$, there is at least one K_j, $j = 1, 2, 3, \ldots, t-1$ which is not a prime factor of N and which can be decomposed into at least two factors. Thus, noting that the final legislature becomes $K_{n+2} = 1$ and that this is brought by a factor decomposition, $x^*_{n+1,1}$ is multiplied by $\frac{1}{2}(\frac{K_{n+1}}{K_{n+2}} + 1)$. Then we obtain

$$
\begin{aligned}
x^*_{n+2,1} &= x^*_{n+1,1} \cdot \frac{1}{2}\left(\frac{K_{n+1}}{K_{n+2}} + 1\right) \\
&= \frac{1}{2}\left(\frac{K_{n+1}}{K_{n+2}} + 1\right)\frac{1}{2}\left(\frac{K_n}{K_{n+1}} + 1\right)\frac{1}{2}\left(\frac{K_{n-1}}{K_n} + 1\right)\frac{1}{2}\left(\frac{K_{n-2}}{K_{n-1}} + 1\right)\cdots \\
&\quad \cdots \frac{1}{2}\left(\frac{K_1}{K_2} + 1\right)\frac{1}{2}\left(\frac{N}{K_1} + 1\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\right)^{n+1}\frac{1}{K_{n+1}K_n \cdots K_2K_1}(K_{n+1}+1)(K_{n+1}+K_n)(K_{n-1}+K_{n-2})\cdots \\
&\quad \cdots (K_2+K_1)(K_1+N)
\end{aligned}
$$

This formula is (1) at $i = n+1$ exactly. □

Lemma 2 is including the case of the most extremely liberal gerrymandering with $t+1$ decision levels. Of course, we can also obtain an analogous result for the most extremely conservative gerrymandering. Additionally, from the proof of Lemma 2, (1) is also indicating the policy position in the direct democracy at $i = 0$. Concerning the final policy position, the direct democracy can be viewed as a special case of a hierarchical political system.

Incidentally, we can use the result of Lemma 2 to explain the partisan voters effect like one of Gilligan & Matsusaka's applications: how many partisan voters are needed at least in order to implement their favorite policy in the legislature at the top-level by gerrymandering.

In our model, we rearrange all voters to those who are supporting a policy of $x_i \in \{0, 1\}$, namely the set of voters' ideal points is $\mathcal{N} = \{0, 0, 0, \ldots, 0, 1, 1, 1, \ldots, 1\}$, where the population of voters is still N with $t+1$ prime factors. Let voters who prefer 0 to 1 be called partisan. Then, how many partisan voters are needed at least? This problem can be solved backwardly. Note that only one representative is at the $t+1$-th decision level, that is $K_{t+1} = 1$. First, one of partisans need to become the final representative to implement their favorite policy. Second, since the partisan final representative is elected from legislature t at the $t+1$-th decision level, they need a majority of the legislators in legislature t, that is at least $\frac{\frac{K_t}{K_{t+1}}+1}{2} = \frac{K_t+1}{2}$ electors because of $K_{t+1} = 1$. Third, since each representative of legislature t at the $t+1$-th decision level is elected in a district in legislature $t-1$ at the t-th decision level, partisans need a majority of representatives in a majority of districts in legislature $t-1$, that is at least $\frac{\frac{K_{t-1}}{K_t}+1}{2}$ representatives in at least $\frac{K_t+1}{2}$ districts on the t-th decision level, consequently they need $\frac{\frac{K_{t-1}}{K_t}+1}{2}\frac{K_t+1}{2}$ representatives in legislature $t-1$ to have a majority in legislature t. Forth, since each representative in legislature $t-1$ at the t-th decision level is elected in a district in legislature $t-2$ at the $t-1$-th decision level, partisans need a majority of representatives in a majority of districts in legislature $t-3$, that is at least $\frac{\frac{K_{t-2}}{K_{t-1}}+1}{2}$ representatives in at least $\frac{\frac{K_{k-1}}{K_t}+1}{2}\frac{K_t+1}{2}$ districts on the $t-1$-th decision level, consequently they need $\frac{\frac{K_{t-2}}{K_{t-1}}+1}{2}\frac{\frac{K_{t-1}}{K_t}+1}{2}\frac{K_t+1}{2}$ representatives in legislature $t-2$ to have a majority in legislature $t-1$. The same logic is continuing until the voters level. Then we can obtain the same formula as (1) by using the same transformation as in the proof of Lemma 2. Thus the next corollary can be obtained:

Corollary 1. *The minimum population of partisans in voters is*

$$\frac{1}{2}\left(\frac{1}{2}\right)^t \frac{1}{K_t K_{t-1}\cdots K_2 K_1}(K_t+1)(K_t+K_{t-1})(K_{t-1}+K_{t-2})\cdots(K_2+K_1)(K_1+N)$$

in order that the final policy becomes partisans' favorite policy by their gerrymandering in the hierarchical system with $t+1$ decision levels.

This corollary can also be obtained by applying the "crack and pack" method at each level. As Gilligan & Matsusaka is mentioning "majority-minority", Lemma 2 and Corollary 1 require strictly optimal gerrymandering for minorities to implement a policy they prefer. To put it another way, if a liberal extremist has the authority to organize all districts in each level and if liberal voters are minorities in all voters, in any levels the extremist should not make super-majority minorities districts where the minorities have super-majorities in a few districts. Even if it might seem partially good for minorities to make a few districts composed of super-majority minorities, particularly if the final policy is decided in proportion to the number of representatives at the $t+1$-th decision level, it actually prevents the minorities from their favorite policy even in the system with the multiple decision levels where they need fewer liberal voters than in the single level legislature by gerrymandering.

In Lemma 2, (1) can be obtained as the final policy being the nearest to the extremist in the left side. From this result, as the characterization of the formula, the next proposition says the monotonicity of alienation in the hierarchical legislatures.

Proposition 1. *The final policy is getting farther away from the median of voters monotonically as the number of hierarchical stages increases.*

Proof. Let $N = a_1 \cdot a_2 \cdot a_3 \ldots \cdot a_t \cdot a_{t+1}$ where all of a_i, $(i = 1, \ldots, t+1)$, are prime factors of N. From Lemma 1, $N = \frac{N}{K_1} \cdot \frac{K_1}{K_2} \cdot \frac{K_2}{K_3} \cdot \ldots \cdot \frac{K_{t-1}}{K_t} \cdot \frac{K_t}{K_{t+1}}$, noting that $K_{t+1} = 1$, in the case of maximum number of hierarchical levels, $t+1$. Without loss of generality, we can correspond $\frac{N}{K_1}, \frac{K_1}{K_2}, \frac{K_2}{K_3}, \ldots, \frac{K_{t-1}}{K_t}, \frac{K_t}{K_{t+1}}$ to $a_1, a_2, a_3, \ldots, a_t, a_{t+1}$, respectively. Then we obtain $K_1 = \frac{N}{a_1}$, $K_2 = \frac{N}{a_1 a_2}$, $K_3 = \frac{N}{a_1 a_2 a_3}$, $\ldots$, $K_t = \frac{N}{a_1 a_2 a_3 \ldots a_t}$, $K_{t+1} = \frac{N}{a_1 a_2 a_3 \ldots a_t a_{t+1}} = 1$.

In the case where there are $i+1$, $(i = 1, \ldots, t)$, decision levels, $N = b_1 b_2 \ldots b_{i+1}$ where each b_j, $(j = 1, \ldots, i)$, is a product of some prime factors of N. By using the same method as the above, we obtain $K_1 = \frac{N}{b_1}$, $K_2 = \frac{N}{b_1 b_2}$, $K_3 = \frac{N}{b_1 b_2 b_3}$, $\ldots$, $K_i = \frac{N}{b_1 b_2 b_3 \ldots b_i}$, $K_{i+1} = \frac{N}{b_1 b_2 b_3 \ldots b_t b_{i+1}} = 1$.

Here, when the number of total levels are i, we choose any two factors in $\{b_1, b_2, \ldots, b_{i+1}\}$ and make the product, for simplicity we choose the last two factors, $b_i b_{i+1}$. Then we get

$K_1 = \frac{N}{b_1}$, $K_2 = \frac{N}{b_1 b_2}$, $K_3 = \frac{N}{b_1 b_2 b_3}$, ..., $K_{i-1} = \frac{N}{b_1 b_2 b_3 \dots b_{i-1}}$, $K'_i = \frac{N}{b_1 b_2 b_3 \dots b_i b_{i+1}} = 1$. Comparing the case of i levels with the case of $i+1$, each of K_1 through K_{i-1} is identical.

We calculate the ratio of $x^*_{i+1,1}$ and $x^*_{i+2,1}$.

$$\begin{aligned}
\frac{x^*_{i,1}}{x^*_{i+1,1}} &= \frac{\frac{1}{2}\left(\frac{1}{2}\right)^{i-1} \frac{1}{K_{i-1}K_{i-2}\cdots K_2 K_1}(K_{i-1}+1)(K_{i-1}+K_{i-2})(K_{i-2}+K_{i-3})\cdots(K_2+K_1)(K_1+N)}{\frac{1}{2}\left(\frac{1}{2}\right)^{i} \frac{1}{K_i K_{i-1}\cdots K_2 K_1}(K_i+1)(K_i+K_{i-1})(K_{i-1}+K_{i-2})\cdots(K_2+K_1)(K_1+N)} \\
&= \frac{K_{i-1}+1}{\frac{1}{2}\frac{1}{K_i}(K_i+1)(K_i+K_{i-1})} \\
&= \frac{2K_i(K_{i-1}+1)}{(K_i+1)(K_i+K_{i-1})}
\end{aligned}$$

Here, since $\forall i \in \{1, 2, 3, \dots, t\}$, $K_{i-1} \geq K_i \geq 1$, $2K_i(K_{i-1}+1) - (K_i+1)(K_i+K_{i-1}) = (K_i - 1)(K_{i-1} - K_i) \geq 0$, so that the numerator is always larger than the denominator. Thus $x^*_{i,1} \geq x^*_{i+1,1}$. □

For a given number of voters N and a given number of levels t we determine the number of districts $K_1, \dots, K_t$ that may result in the most biased outcome. Noting that $K_0 = N$, the next proposition can be obtained.

Proposition 2. *In the case of N voters and $t+1$ decision levels for which $\sqrt[t+1]{N}$ is an integer, the multi-level districting given by*

$$\forall i = 1, \dots, t, t+1 : K_i = N^{\frac{t-(i-1)}{t+1}} \tag{2}$$

admits the largest possible bias.

Proof. Determining maximum bias is equivalent with minimizing $x^*_{t+1,1}/N$ with respect to $K_1, \dots, K_t$. In the next formula, noting that $N = K_0$,

$$\frac{x^*_{t+1,1}}{N} = \left(\frac{1}{2}\right)^{t+1} \frac{1}{K_t K_{t-1} \cdots K_2 K_1 K_0}(K_t+1)(K_t+K_{t-1})(K_{t-1}+K_{t-2})\cdots(K_2+K_1)(K_1+K_0),$$

we multiply $1/K_0, 1/K_1, \dots, 1/K_t$ to the terms from the last in the reverse order. Hence, we have to minimize expression

$$\left(\frac{1}{2}\right)^{t+1}\left(1+\frac{1}{K_t}\right)\left(1+\frac{K_t}{K_{t-1}}\right)\cdots\left(1+\frac{K_2}{K_1}\right)\left(1+\frac{K_1}{K_0}\right) \tag{3}$$

yielding first-order conditions equivalent with

$$0 = -\frac{K_{i+1}}{K_i^2}\left(1+\frac{K_i}{K_{i-1}}\right) + \left(1+\frac{K_{i+1}}{K_i}\right)\frac{1}{K_{i-1}}$$

for all $i = 1, \dots, t$; from which we can obtain by simple rearrangements

$$K_{i+1}K_{i-1} = K_i^2 \tag{4}$$

for all $i = 1, \ldots, t$, where $K_{t+1} = 1$. It can be verified that the first-order conditions determine the minimum value for expression (3).

We claim that

$$K_{t-(i-1)} = (K_{t-i})^{\frac{i}{i+1}} \tag{5}$$

for all $i = 1, \ldots, t$, which we prove by induction. Clearly, our claim holds true for $i = 1$ by equation (4). Assume that (5) is valid for i and we show that it also holds true for $i+1$. From (4) we have

$$K_{t-i+1}K_{t-i-1} = (K_{t-i})^2\,,$$

and by employing our induction hypothesis we get

$$(K_{t-i})^{\frac{i}{i+1}}\, K_{t-i-1} = (K_{t-i})^2 \quad \Leftrightarrow \quad K_{t-i} = (K_{t-i-1})^{\frac{i+1}{i+2}}\,,$$

which is what we wanted to show.

Finally, by employing (5) recursively we obtain the statement of our proposition. □

Assuming in Proposition 2 that $\sqrt[t+1]{N}$ is an integer, appears to be too restrictive. [4] Clearly, for arbitrary combinations of N and t the sequence $(K_i)_{i=1}^{t+1}$ given by equation (2) is typically a non-integer valued sequence and does not determine legitimate district sizes. However, it can be checked that the largest possible bias will approach the bias corresponding to the sequence given by (2) if N approaches infinity, while t remains fixed.

In fact Proposition 2 describes the hierarchical structure of a multi-level gerrymandering. Our next corollary determines the respective worst case bias.

Corollary 2. *If the number of levels t is fixed, then as the number of voters N tends to infinity worst-case bias approaches*

$$\left|F\left(x^*_{t+1,1}\right) - F\left(x^*_{POP}\right)\right| = \frac{1}{2} - \left(\frac{1}{2}\right)^{t+1}$$

Proof. From Proposition 2, we get

$$\frac{K_{i+1}}{K_i} = \frac{N^{\frac{t-i}{t+1}}}{N^{\frac{t-(i-1)}{t+1}}} = \left(\frac{1}{N}\right)^{\frac{1}{t+1}}. \tag{6}$$

By substituting (6) into (3) and letting N tend to infinity, we obtain our corollary. □

Now, from Lemma 1 the maximum number of levels t gets larger as N becomes larger. Then the maximum bias approaches its highest possible level, which we state in the next corollary.

Corollary 3. *As t tends to infinity maximum worst-case bias in case of liberal gerrymandering tends to $x^*_{POP} - x_1$.*

In particular, if party A is at the left end and party B on the right end of the unit interval, then, for instance, party A's most preferred outcome will be the national outcome independently form the voters preferences.

[4]Not to mention, if all prime factors of N are identical, for example $N = a^{t+1}$ where a is a prime factor, $\sqrt[t+1]{N}$ is always integer.

4 Random districting

One may hope that the extent of the worst case bias shown in the previous section will not cause a severe problem because, for instance, random districting (that is, voters are randomly grouped into districts) could solve the problem. However, in this section we show that the results on random districting obtained by Gilligan and Matsusaka (2006) remain valid for our multiple hierarchical model of representative democracy.

First, assume that N is odd, $\lceil N/2 \rceil + 1$ voters have their ideal points at 1 and $\lfloor N/2 \rfloor$ voters have their ideal points at 2.[5] We assume again for simplicity that the number of voters N takes a value such that for given $K_1, \ldots, K_t$ a t-level districting can be carried out in integers.[6] The median voter's ideal point equals 1. However, one can group the voters into equally sized first-level districts such that the ideal point of the first-level median representative equals 2, which means that we have more first-level districts with a median representative at 2 than at 1. Now at the second-level one can also construct a districting having more representatives with ideal points 2 than 1. We can proceed in the same way until we arrive at the top level, which then has a median representative with ideal point 2. Since these type of districting emerge with positive probability the expected ideal point will exceed the voters' median point.

In the described example voters' ideal points are a bit skewed. The next Proposition, which is analogous to Proposition 2 of Gilligan and Matsusaka (2006), investigates the cases of symmetric and up-wards skewed distribution of voters' ideal points.

Proposition 3. *Assuming that each districting is equally probable, the expected bias of random districting*

1. *is zero if the voters' ideal points are symmetrically distributed around their median, and*

2. *is biased up-wards if the voters' ideal points are skewed up-wards.*

Proof. Assume that the voters' ideal points are ordered increasingly, i.e. $x_1^* \leq x_2^* \leq \ldots \leq x_N^*$. Let $M = (N+1)/2$. We shall denote by the $p(x_i)$ the probability that voter i becomes the top-level representative, that is, policy-maker or legislator.

We start with proving point 1. Because of the symmetric setting we must have $p(x_{M-i}) = p(x_{M+i})$ and $x_M - x_{M-i} = x_{M+i} - x_M$ for all $i = 0, \ldots, M-1$. Hence,

$$E\left(x_{LEG}^*\right) = \sum_{i=1}^{N} p(x_i) = \sum_{i=1}^{M-1} p(x_{M-i})x_{M-i} + p(x_M)x_M + \sum_{i=1}^{M-1} p(x_{M+i})x_{M+i} = x_M. \qquad (7)$$

For establishing point 2, we just have to replace $x_M - x_{M-i} = x_{M+i} - x_M$ with $x_M - x_{M-i} \leq x_{M+i} - x_M$ for all $i = 0, \ldots, M-1$, which holds since the distribution of ideal points is up-wards skewed, in equation (7). □

[5] This example is an extension of an example by Gilligan and Matsusaka (2006, p. 387).

[6] Hence, each district has a uniquely determined median voter.

5 A moderate district maker

We have considered the crack and pack method in the case where a voter with very lower-value number who is an extremist in the left wing and who is only a district maker wants to implement a policy as low as possible, so far. However an extremist is not always a district maker, that is another voter with a moderate political position around the voters median can be a district maker. Then she will consider how all districts should be organized to implement her favorite policy or how she should behave as a district maker like the extremist in the previous sections. Is her favorite policy implementable? Yes, we can organize districts to implement it by generalizing the crack and pack method of the previous sections, hereafter it is called the generalized crack and pack method. Noting that the interval we have to consider is $\{1, 2, \ldots, \frac{N+1}{2}\}$ because of symmetry, we will begin to generalize the crack and pack method. We can define the district numbers which $x^*_{t+1,1}$ belongs to at each decision level in the crack and pack method as

$$m_1 \equiv \frac{x^*_{t+1,1}}{\frac{1}{2}(\frac{N}{K_1}+1)} = \frac{1}{2}(K_t+1)\frac{1}{2}\left(\frac{K_{t-1}}{K_t}+1\right)\frac{1}{2}\left(\frac{K_{t-2}}{K_{t-1}}+1\right)\ldots\frac{1}{2}\left(\frac{K_1}{K_2}+1\right),$$

$m_2 \equiv \frac{m_1}{\frac{1}{2}(\frac{K_1}{K_2}+1)}$, $m_3 \equiv \frac{m_2}{\frac{1}{2}(\frac{K_2}{K_3}+1)}$, $\ldots$, $m_t \equiv \frac{m_{t-1}}{\frac{1}{2}(\frac{K_{t-1}}{K_t}+1)}$, and $m_{t+1} \equiv \frac{m_t}{\frac{1}{2}(K_t+1)}$, since there are minimum majority voters or representatives with lower value ideal points that are $\frac{1}{2}(\frac{K_{i-1}}{K_i}+1)$, $i \in \{1, 2, 3, \ldots, t, t+1\}$ before $x^*_{t+1,1}$ at each decision level. Then we have the next lemma.

Lemma 3. *Any voter between $x^*_{t+1,1}$ and the median of all voters is electable to a district representative to the second decision level.*

Proof. In the crack and pack method, from the definition of m_i, the district number where $x^*_{t+1,1}$ is included at the first decision level is m_1. Then maximum minority voters with higher-value number that are $\frac{1}{2}(\frac{N}{K_1}-1)$ voters are inserted in the first district through the m_1-th. Those are showed in Table 6.

Table 6: Each district by the Crack and Pack method at the first decision level.

	The position of each voter							
District number	1	2	$\ldots$	$\frac{1}{2}(\frac{N}{K_1}+1)$ = median	$\frac{1}{2}(\frac{N}{K_1}+1)+1$	$\frac{1}{2}(\frac{N}{K_1}+1)+2$	$\ldots$	$\frac{1}{2}(\frac{N}{K_1}+1)+\frac{1}{2}(\frac{N}{K_1}-1)=\frac{N}{K_1}$
1	1	2	$\ldots$	A	N	$N-1$	$\ldots$	$N-B+1$
2	$A+1$	$A+2$	$\ldots$	$2A$	$N-B$	$N-B-1$	$\ldots$	$N-2B+1$
$\vdots$		$\vdots$				$\vdots$		
m_1-1	$(m_1-2)A+1$	$(m_1-2)A+2$	$\ldots$	$(m_1-1)A$	$N-(m_1-2)B$	$N-(m_1-2)B-1$	$\ldots$	$N-(m_1-1)B+1$
m_1	$(m_1-1)A+1$	$(m_1-1)A+2$	$\ldots$	m_1A	m_1A+1	m_1A+2	$\ldots$	m_1A+B
m_1+1	m_1A+B+1	m_1A+B+2	$\ldots$	$(m_1+1)A+B$	$(m_1+1)A+B+1$	$(m_1+1)A+B+2$	$\ldots$	$(m_1+1)A+2B$
$\vdots$		$\vdots$				$\vdots$		
K_1-1	$(K_1-2)A+(K_1-m_1-1)B+1$	$(K_1-2)A+(K_1-m_1-1)B+2$	$\ldots$	$(K_1-1)A+(K_1-m_1-1)B$	$(K_1-1)A+(K_1-m_1-1)B+1$	$(K_1-1)A+(K_1-m_1-1)B+2$	$\ldots$	$(K_1-1)A+(K_1-m_1)B$
K_1	$(K_1-1)A+(K_1-m_1)B+1$	$(K_1-1)A+(K_1-m_1)B+2$	$\ldots$	$K_1A+(K_1-m_1)B$	$K_1A+(K_1-m_1)B+1$	$K_1A+(K_1-m_1)B+2$	$\ldots$	$K_1A+(K_1-m_1+1)B$

$A \equiv \frac{1}{2}(\frac{N}{K_1}+1)$, $B \equiv \frac{1}{2}(\frac{N}{K_1}-1)$

Now we focus on the median of each district, especially the m_1-th district through the district including the median of all voters after the following manipulations. Note that maximum minority voters with higher-value are already inserted into only the first through the $m_1 - 1$-th districts and that voters are consecutive numbers in the m_1-th district through the K_1-th. Here, we remove the last-position voter with higher-value number in the $m_1 - 1$-th district, and instead, insert the first-position voter with lowest-value in the m_1-th district into the last-position. Then we slide back all voters between the m_1-th and the K_1-th districts by one position, and insert the removed voter from the $m_1 - 1$-th district into the last position that is vacant by this slide in the K_1-th district. By these manipulations, medians in the m_1-th district through the K_1-th are slid and changed by each voter with one higher-value. By repeating the manipulations one by one, the positions of voters in the $m_1 - 1$-th district become Table 7. Noting that there are $\frac{N}{K_1}$ voters in each district and noting that all district medians of the m_1-th district through the K_1-th are slid one by one, we have one cycle by repeatedly sliding $\frac{N}{K_1} - 1$ voters back, that is all voters between $x^*_{t+1,1}$ and $\frac{N+1}{2}$ appear as district medians. Noting that $\frac{1}{2}(\frac{N}{K_1} - 1)$ voters with higher value are inserted in the first district through $m_1 - 1$-th, by changing voters in $(\frac{N}{K_1} - 1)/(\frac{1}{2}(\frac{N}{K_1} - 1)) = 2$ districts which are the $m_1 - 1$-th and the $m_1 - 2$-th, we have this one cycle. Then, let the district number including the median of all voters be m in no sliding back, [7] we can line district medians up consecutively from the m_1-th district through the m-th and have each district median of those districts as Table 8. From the definition of m, $\frac{m_1}{2}(\frac{N}{K_1}+1)+(m-m_1)\frac{N}{K_1} \leq \frac{N+1}{2} \leq \frac{m_1}{2}(\frac{N}{K_1}+1)+(m-m_1+1)\frac{N}{K_1}-1$ in the last column of the table. Noting that the numbers from $\frac{m_1}{2}(\frac{N}{K_1}+1)$ are lined up in each columns of Table 8 in consecutive ascending order, at least we can place voters between $\frac{m_1}{2}(\frac{N}{K_1}+1)$ and $\frac{N+1}{2}$ at district median positions of the m_1-th district through the m-th. Thus all voters between $\frac{m_1}{2}(\frac{N}{K_1}+1) = x^*_{t+1,1}$ and $\frac{N+1}{2}$ can be districts median by the above manipulations, in other words those voters are electable to district representatives to the second decision level. $\square$

Table 7: The positions of voters in the $m_1 - 1$-th district at the first decision level.

	Voters' positions					
Slide back	…	median	median $+1$	median $+2$	…	median$+\frac{1}{2}(\frac{N}{K_1}-1) = \frac{N}{K_1}$
0		$\frac{m_1-1}{2}(\frac{N}{K_1}+1)$	$N - \frac{m_1-2}{2}(\frac{N}{K_1}-1)$	$N - \frac{m_1-2}{2}(\frac{N}{K_1}-1) - 1$		$N - \frac{m_1-1}{2}(\frac{N}{K_1}-1) + 1$
1		$\frac{m_1-1}{2}(\frac{N}{K_1}+1)$	$\frac{m_1-1}{2}(\frac{N}{K_1}+1) + 1$	$N - \frac{m_1-2}{2}(\frac{N}{K_1}-1)$		$N - \frac{m_1-1}{2}(\frac{N}{K_1}-1) + 2$
2	…	$\frac{m_1-1}{2}(\frac{N}{K_1}+1)$	$\frac{m_1-1}{2}(\frac{N}{K_1}+1) + 1$	$\frac{m_1-1}{2}(\frac{N}{K_1}+1) + 2$	…	$N - \frac{m_1-1}{2}(\frac{N}{K_1}-1) + 3$
$\vdots$			$\vdots$			$\vdots$
$\frac{1}{2}(\frac{N}{K_1}-1)$		$\frac{m_1-1}{2}(\frac{N}{K_1}+1)$	$\frac{m_1-1}{2}(\frac{N}{K_1}+1) + 1$	$\frac{m_1-1}{2}(\frac{N}{K_1}+1) + 2$		$\frac{m_1}{2}(\frac{N}{K_1}+1) - 1.$

[7] In this case, $m = \lceil x \rceil$ such that

$$\frac{1}{2}\left(\frac{N}{K_1}+1\right)(m_1-1) + \frac{N}{K_1}(x-m_1+1) = \frac{N+1}{2}.$$

Table 8: Each district median at the first decision level.

Slide back	District median of the m_1-th dist.	m_1+1-th dist.	m_1+2-th dist.	$\dots$	m-th dist.	$\dots$
0	$\frac{m_1}{2}(\frac{N}{K_1}+1)$	$\frac{m_1}{2}(\frac{N}{K_1}+1)+\frac{N}{K_1}$	$\frac{m_1}{2}(\frac{N}{K_1}+1)+\frac{2N}{K_1}$		$\frac{m_1}{2}(\frac{N}{K_1}+1)+(m-m_1)\frac{N}{K_1}$	$\dots$
1	$\frac{m_1}{2}(\frac{N}{K_1}+1)+1$	$\frac{m_1}{2}(\frac{N}{K_1}+1)+\frac{N}{K_1}+1$	$\frac{m_1}{2}(\frac{N}{K_1}+1)+\frac{2N}{K_1}+1$			
2	$\frac{m_1}{2}(\frac{N}{K_1}+1)+2$	$\frac{m_1}{2}(\frac{N}{K_1}+1)+\frac{N}{K_1}+2$	$\frac{m_1}{2}(\frac{N}{K_1}+1)+\frac{2N}{K_1}+2$	$\dots$	$\vdots$	
$\vdots$		$\vdots$				
$\frac{N}{K_1}-1$	$\frac{m_1}{2}(\frac{N}{K_1}+1)+\frac{N}{K_1}-1$	$\frac{m_1}{2}(\frac{N}{K_1}+1)+\frac{2N}{K_1}-1$	$\frac{m_1}{2}(\frac{N}{K_1}+1)+\frac{3N}{K_1}-1$		$\frac{m_1}{2}(\frac{N}{K_1}+1)+(m-m_1+1)\frac{N}{K_1}-1$	$\dots$

It seems to any moderate voters can be elected as the final representative by also applying this lemma at higher decision levels repeatedly. However, unfortunately, this lemma does not guarantee that those voters are always electable at the first through the final decision level by this method from only left wing. See the latter half of Example 2. There are some cases where we need to combine the sliding back from both left and right wings. That is too complicated a little bit. If we elect any moderate voters by the generalized crack and pack method from only left side wing, we need to slide back more voters until the voters median come back at the district median position of the median district. What has to be noticed is $\frac{K_1+1}{2} \leq m$, [8] that is the voters median is not a district median between the m_1-th and $\frac{K_1+1}{2}$-th districts by only one cycle in Table 8. Because the voters median is placed at the voters median position when all voters with higher-value in the first district through the m_1-1-th are slid back that is all N voters are lined up in ascending order, which is called full slide back, hereafter. In the other cases, the voters median is at the back of the voters median position. Thus if we want a voter between $x^*_{t+1,1}$ and the voters median to be electable to the final representative, one more condition is needed.

Lemma 4. *Any voter between $x^*_{t+1,1}$ and the median of all voters is electable to a district representative of the m_2-th district through the $\frac{K_2+1}{2}$-th to the third decision level at the second decision level.*

Proof. Noting that $x^*_{t+1,1} = \frac{m_1}{2}(\frac{N}{K_1}+1)$ and let $V_1 \equiv \frac{m_1-1}{2}(\frac{N}{K_1}-1)$, then we have Table 9 of the full slide back case. Focusing on the last row V_1 in Table 9, starting from $x^*_{t+1,1}+V_1 = (m_1-1)\frac{N}{K_1}+\frac{1}{2}(\frac{N}{K_1}+1)$ in the first column, the voters median $x^*_{t+1,1}+(\frac{K_1+1}{2}-m_1)\frac{N}{K_1}+V_1 = \frac{N+1}{2}$ is in the last column since all voters are lined up in ascending order. In other words, by sliding back by $\frac{m_1-1}{2}(\frac{N}{K_1}-1)$ voters at the first decision level, the voters median takes the district

[8] When the crack and pack method is used, the voters median is at the back of the voters median position since voters with higher-value are added in the front districts. Thus $\frac{K_1+1}{2} \leq m$. On the other hand, when all voters are lined up and districted in the ascending order, the voters median is placed at the district median of the median district that is $\frac{K_1+1}{2} = m$. Thus the voters median comes back the median position as sliding back voters one by one.

Table 9: Each district median of the m_1-th through the $\frac{K_1+1}{2}$-th at the first decision level.

Slide back	Median voter of the m_1-th dist.	m_1+1-th dist.	m_1+2-th dist.	...	$\frac{K_1+1}{2}$-th dist.
0	$x^*_{t+1,1}$	$x^*_{t+1,1}+\frac{N}{K_1}$	$x^*_{t+1,1}+\frac{2N}{K_1}$	...	$x^*_{t+1,1}+(\frac{K_1+1}{2}-m_1)\frac{N}{K_1}$
1	$x^*_{t+1,1}+1$	$x^*_{t+1,1}+\frac{N}{K_1}+1$	$x^*_{t+1,1}+\frac{2N}{K_1}+1$		
2	$x^*_{t+1,1}+2$	$x^*_{t+1,1}+\frac{N}{K_1}+2$	$x^*_{t+1,1}+\frac{2N}{K_1}+2$		$\vdots$
$\vdots$		$\vdots$			
$\frac{N}{K_1}-1$	$x^*_{t+1,1}+\frac{N}{K_1}-1$	$x^*_{t+1,1}+\frac{N}{K_1}+\frac{N}{K_1}-1$	$x^*_{t+1,1}+\frac{2N}{K_1}+\frac{N}{K_1}-1$		$x^*_{t+1,1}+(\frac{K_1+1}{2}-m_1)\frac{N}{K_1}+\frac{N}{K_1}-1$
$\vdots$		$\vdots$			$\vdots$
V_1	$x^*_{t+1,1}+V_1$	$x^*_{t+1,1}+\frac{N}{K_1}+V_1$	$x^*_{t+1,1}+\frac{2N}{K_1}+V_1$	...	$x^*_{t+1,1}+(\frac{K_1+1}{2}-m_1)\frac{N}{K_1}+V_1$
Group number	m_1	m_1+1	m_1+2	...	$\frac{K_1+1}{2}$

median position of the median district. Note that there are some repeated numbers in Table 9 since we slide back voters more than one cycle case in Table 8. [9]

Here we name medians of the m_1-th district Group m_1, those of the m_1+1-th Group m_1+1 and so on, and in the end that of $\frac{K_1+1}{2}$-th district Group $\frac{K_1+1}{2}$. We show how many voters need to be slid back to elect a voter to the second level. We need to choose a row $i \in \{0, 1, \ldots, V_1\}$ including a voter whom we want to elect as a district representative to the second decision level. Then if the elected voter is in Group m_1, we do not have to slide back any position at the second decision level since she is already at the same position as $x^*_{t+1,1}$ when sliding back 0 and becomes a district median at the second decision level. If the elected voter is in Group m_1+1, she shifts out of one position of $x^*_{t+1,1}$ in the sliding back 0. If in Group m_1+2, out of two positions, if in Group m_1+3, out of three positions, and so on. Thus we have to slide back the elected voter appropriately at the second level to elect her as a district representative to the third decision level.

For simplicity, we renumber elected representatives to the second level as $\{1, 2, 3, \ldots, m_1, \ldots, K_1\}$. Let $m_2 \equiv \dfrac{m_1}{\frac{1}{2}(\frac{K_1}{K_2}+1)}$. When sliding back by 0 position at the second level, each representative of Group m_1 becomes the district median of the m_2-th district. When sliding back by 1 position at the second level, each representative of Group m_1+1 becomes the district median of the m_2-th district, and so on. Let $V_2 \equiv \frac{m_2-1}{2}(\frac{K_1}{K_2}-1)$ like V_1 at the first decision level. Sliding back each district representative one by one, we have each district median in Table 10 at the

[9] In Table 9, the conditions of voters consecutively lined up without repetitions are $V_1+1=\frac{N}{K_1}$ that is $m_1=3$ or $\frac{N}{K_1}=1$. The latter condition is the case where each district is composed of one voter. On the other hand, when $V_1+1<\frac{N}{K_1}$ that is $m_1<3$ and $\frac{N}{K_1}>1$, voters are not lined up consecutively in Table 9. However if $m_1=2$, since there is only one district before the m_1-th and $V_1=\frac{1}{2}(\frac{N}{K_1}-1)$, we can slide back all voters only in the first column. If $m_1=1$, total district is one, so that the voters median is the district median. When $m_1>3$ and $\frac{N}{K_1}>1$, there are some repetition.

second decision level as well as Table 9. Note that m_1 is a representative of Group m_1 in

Table 10: Each district median of the m_2-th through the $\frac{K_2+1}{2}$-th at the second decision level.

Slide back	Median voter of the m_2-th dist.	m_2+1-th dist.	m_2+2-th dist.	$\dots$	$\frac{K_2+1}{2}$-th dist.
0	m_1	$m_1+\frac{K_1}{K_2}$	$m_1+\frac{2K_1}{K_2}$	$\dots$	$m_1+(\frac{K_2+1}{2}-m_2)\frac{K_1}{K_2}$
1	m_1+1	$m_1+\frac{K_1}{K_2}+1$	$m_1+\frac{2K_1}{K_2}+1$		
2	m_1+2	$m_1+\frac{K_1}{K_2}+2$	$m_1+\frac{2K_1}{K_2}+2$		$\vdots$
$\vdots$		$\vdots$			
$\frac{K_1}{K_2}-1$	$m_1+\frac{K_1}{K_2}-1$	$m_1+\frac{K_1}{K_2}+\frac{K_1}{K_2}-1$	$m_1+\frac{2K_1}{K_2}+\frac{K_1}{K_2}-1$		$m_1+(\frac{K_2+1}{2}-m_2)\frac{K_1}{K_2}+\frac{K_1}{K_2}-1$
$\vdots$		$\vdots$			$\vdots$
V_2	m_1+V_2	$m_1+\frac{K_1}{K_2}+V_2$	$m_1+\frac{2K_1}{K_2}+V_2$	$\dots$	$m_1+(\frac{K_2+1}{2}-m_2)\frac{K_1}{K_2}+V_2$
Group number	m_2	m_2+1	m_2+2	$\dots$	$\frac{K_2+1}{2}$

Table 9, for example if the row of the slide back 2 is chosen at the first level, $m_1 = x^*_{t+1,1}+2$. Focusing on the last column of the last row V_2 in Table 10,

$$\begin{aligned} & m_1 + \left(\frac{K_2+1}{2} - m_2\right)\frac{K_1}{K_2} + V_2 \\ = \; & m_1 + \left(\frac{K_2+1}{2} - \frac{m_1}{\frac{1}{2}(\frac{K_1}{K_2}+1)}\right)\frac{K_1}{K_2} + \frac{\frac{m_1}{\frac{1}{2}(\frac{K_1}{K_2}+1)}-1}{2}\left(\frac{K_1}{K_2}-1\right) = \frac{K_1+1}{2}, \end{aligned}$$

that means a representative of Group $\frac{K_1+1}{2}$ at the first level, when all representatives are lined up in ascending order as well as at the first level. In other words, since the row of V_2 is the full sliding back, a representative of Group $\frac{K_1+1}{2}$ becomes the district median of the median district at the second decision level. Thus elected district representatives at the first level become the district representatives between m_2-th and $\frac{K_2+1}{2}$-th district to the third decision level at the second decision level. □

Noting that each decision level has the same structure, all elected representatives are renumbered from one to K_i, $i = \{1, 2, 3, \dots, t\}$ at each level. Then, at the upper decision levels than the second, if district representatives are between m_i and $\frac{K_i+1}{2}$ at one-lower decision level that is the i-th level, Lemma 4 can also be applicable at the $i+1$ level. Thus we have the next lemma.

Lemma 5. *If a renumbered representative elected at the i-th decision level is a representative between m_i and $\frac{K_i+1}{2}$ at the $i+1$-th decision level where there are K_i representatives, $i \in \{1, 2, 3, \dots, t\}$, she is electable to a district representative between the m_{i+1}-th district and the $\frac{K_{i+1}+1}{2}$-th at the $i+1$-th decision level.*

Proof. The proof of this lemma becomes the same as the proof of Lemma 4 by replacing N with K_i, K_1 with K_{i+1}, m_1 with m_i, and m_2 with m_{i+1}. □

Applying Lemma 5 repeatedly, we have the below proposition in finally.

Proposition 4. *Any voter between* $x^*_{t+1,1}$ *and the median of all voters* $\frac{N+1}{2}$ *can be electable to the final representative.*

Proof. Noting that $K_1 \geq K_2 \geq K_3 \geq \ldots \geq K_t \geq K_{t+1} = 1$, since $\frac{1}{2}(\frac{K_i}{K_{i+1}} + 1) \geq 1$, we have

$$m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_t \geq m_{t+1} = 1$$

from the definition of m_i, and we have

$$\frac{N+1}{2} \geq \frac{K_1+1}{2} \geq \frac{K_2+1}{2} \geq \ldots \geq \frac{K_t+1}{2} \geq \frac{K_{t+1}+1}{2} = 1.$$

Thus, by applying the proof of Lemma 5, both m_{i+1} and $\frac{K_{i+1}+1}{2}$ like Table 9 and 10 is shrinking to 1 as $i \to t$. In finally, we can say that a voter we want to elect as the final representative between $x^*_{t+1,1}$ and the voters median $\frac{N+1}{2}$ is elected at the final decision level by the sandwich. □

The next example illustrates the above.

Example 2. Let us consider the example of $N = 27$, $t + 1 = 3$ and the voters set $\mathcal{N} = \{1, 2, 3, \ldots, 25, 26, 27\}$ again. In this case, the median of all voters is 14, and the most extremely liberal voter who is electable to the final representative is $x^*_{3,1} = \frac{1}{2}(3+1) \cdot \frac{1}{2}(\frac{9}{3}+1) \cdot \frac{1}{2}(\frac{27}{9}+1) = 8$ which belongs to the $m_1 = 4$-th district. Thus at the first decision level only medians elected from the 4-th district including voter 8 and from the 5-th district becoming the median of the second level are electable at the first through the final level.

Full slide back case: We just have to slide back voters by zero, one, two and three positions to place voters $8, 9, 10, 11, 12, 13, 14$ at the 4-th and 5-th district median positions since $\frac{m_1-1}{2}(\frac{N}{K_1}-1) = \frac{4-1}{2}(\frac{27}{9}-1) = 3$, then we have Table 11. If we want voter 12 to be elected

Table 11: Placing voters between 8 and 14 at district median positions at the first level.

Slide	District number								
back	1	2	3	4	5	6	7	8	9
0	$\{1,2,27\}$	$\{3,4,26\}$	$\{5,6,25\}$	$\{7,\boxed{8},9\}$	$\{10,\underline{11},12\}$	$\{13,14,15\}$	$\{16,17,18\}$	$\{19,20,21\}$	$\{22,23,24\}$
				↓	↓				
1	$\{1,2,27\}$	$\{3,4,26\}$	$\{5,6,7\}$	$\{8,\underline{9},10\}$	$\{11,\underline{12},13\}$	$\{14,15,16\}$	$\{17,18,19\}$	$\{20,21,22\}$	$\{23,24,25\}$
				↓	↓				
2	$\{1,2,27\}$	$\{3,4,5\}$	$\{6,7,8\}$	$\{9,\underline{10},11\}$	$\{12,\underline{13},14\}$	$\{15,16,17\}$	$\{18,19,20\}$	$\{21,22,23\}$	$\{24,25,26\}$
				↓	↓				
3	$\{1,2,3\}$	$\{4,5,6\}$	$\{7,8,9\}$	$\{10,\underline{11},12\}$	$\{13,\boxed{14},15\}$	$\{16,17,18\}$	$\{19,20,21\}$	$\{22,23,24\}$	$\{25,26,27\}$

to a district representative to the second through the final decision level, we need to slide one positions, then we have the representatives $\{2, 4, 6, 9, 12, 15, 18, 21, 24\}$. At the second decision level, we need to slide back representatives by one position:

Slide back	1	2	3
1	$\{2,4,6\}$	$\boxed{\{9,12,15\}}$	$\{18,21,24\}$,

and we have representatives $\{4,12,21\}$ to the final decision level. In finally, 12 is elected as the final representative.

Next, if we want voter 14 to be elected to a district representative to the second through the final decision level, we need to slide back three positions at the first decision level, then we have the representatives $\{2,5,8,11,14,17,20,23,26\}$. At the second level, we need to slide back representatives by one position for representative 14 to belong to the median district:

Slide back	1	2	3
1	$\{2,5,8\}$	$\boxed{\{11,14,17\}}$	$\{20,23,26\}$.

At final level, there is only one $\{5,14,23\}$, then 14 is elected as the final representative. In this case, voters are slid back fully at each decision level so that all voters are lined up consecutively in ascending order.

In finally, if we want voter 11 to be elected to a district representative to the second through the final decision level, sliding back either zero or three positions are fine. If zero position is chosen, we have the representatives $\{2,4,6,8,11,14,17,20,23\}$, and if three positions are chosen, the representatives are $\{2,5,8,11,14,17,20,23,26\}$. In the former, at the second decision level, representatives need to be slid back by one position, and in the latter, they need to be slid back by zero position:

Slide back				
first level	second level	1	2	3
0	1	$\{2,4,6\}$	$\{8,11,14\}$	$\{17,20,23\}$
1	0	$\{2,5,27\}$	$\{8,11,14\}$	$\{17,20,23\}$.

Since 11 appears as the district median of the fourth and fifth districts in zero and three sliding back, respectively, either one is fine at the first decision level, then we can obtain the same result at the final decision level.

One cycle case: On the other hand, if we slide voters by only one cycle that is sliding back $\frac{N}{K_1}-1=\frac{27}{9}-1=2$ voters, we have Table 12. In this case, the voters median 14 is a district median in zero sliding and is in neither the 4-th and the 5-th districts but in the 6-th. Can voter 14 be electable at all decision levels? No, she is not elected as a district representative at the second level from the left wing. The district representatives to the second level are $\{2,4,6,8,11,14,17,20,23\}$. At the second level, although we can chose either zero or one sliding back,

Slide back	1	2	3
0	$\{2,4,23\}$	$\{6,8,11\}$	$\{14,17,20\}$.
1	$\{2,4,6\}$	$\{8,11,14\}$	$\{17,20,23\}$.

Table 12: Sliding voters back by only one cycle at the first decision level.

Slide back	District number 1	2	3	4	5	6	7	8	9
0	$\{1,2,27\}$	$\{3,4,26\}$	$\{5,6,25\}$	$\{7,\boxed{8},9\}$	$\{10,\underline{11},12\}$	$\{13,\boxed{14},15\}$	$\{16,17,18\}$	$\{19,20,21\}$	$\{22,23,24\}$
				$\downarrow$	$\downarrow$				
1	$\{1,2,27\}$	$\{3,4,26\}$	$\{5,6,7\}$	$\{8,\underline{9},10\}$	$\{11,\underline{12},13\}$	$\{14,15,16\}$	$\{17,18,19\}$	$\{20,21,22\}$	$\{23,24,25\}$
				$\downarrow$	$\downarrow$				
2	$\{1,2,27\}$	$\{3,4,5\}$	$\{6,7,8\}$	$\{9,\underline{10},11\}$	$\{12,\underline{13},14\}$	$\{15,16,17\}$	$\{18,19,20\}$	$\{21,22,23\}$	$\{24,25,26\}$.

14 cannot be any district median. When zero sliding back, $\{4,8,17\}$ are elected at the second level and voter 8 becomes the final representative. When one sliding back, $\{4,11,20\}$ are elected and voter 11 becomes the final representative. Thus in the one cycle sliding, although the voters can be elected as representatives to only the next decision level, they can not always be elected as a representative to after the next level. □

6 Concluding remarks

In this paper we introduced a multiple hierarchical model of representative democracy in the hope that it might result in a "more direct" democracy since the connections between a representative and its voters can be strengthened by the fact that a representative is answerable for less voters as the number of hierarchical levels increases. However, instead of supporting evidence for our hierarchical model we found its increased vulnerability to gerrymandering and policy bias. This might contribute to a formal explanation to why the attempts of implementing hierarchical models of representative democracy in the past failed and contribute to a better understanding of representative democracy.

In future research we plan to investigate random districting further and introduce informational structures into our model such as the verifiability of representatives behavior and the privacy of individual votes.

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