Step-by-step group contests with group-specific public-good prizes*

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Abstract

This study analyzes group contests with group-specific public-good prizes in which the "step-by-step" characteristic is introduced. For example, research groups must expend much effort to publish papers one by one. Once a paper is accepted by a top journal, a group has a major opportunity to receive a new research grant. On the contrary, if the paper is rejected by all journals, group members obtain nothing despite the effort expended. In this study, we focus on this "one or nothing" characteristic and introduce a step function as a group impact function. We then characterize the Nash equilibria. We show the condition of the existence of the Nash equilibrium at which no group member free-rides on the others, showing that the effort levels in the Nash equilibria are less than the optimal effort level for each group. These results are different from those derived from the continuous group impact function.

Keywords: Step-by-step technology, Group contest, Group-specific public good

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1 Introduction

In research fields, especially experimental fields in biology, physics, chemistry, and medicine, groups that carry out research projects are often organized at universities. Group members must expend much effort, with the associated problem of members' free-riding, to publish their papers. Once their findings are published in a top journal, group members obtain benefits, particularly the opportunity to receive a new research grant[1]. On the contrary, if their paper is rejected by all journals, each member obtains nothing despite the effort expended. Thus, all research groups face the problem of "one or nothing" for each of their publications.

Each group tries to publish papers "step-by-step", discretely, by continuing its research. When the grant available is a competitive research grant[2], research groups often compete with other groups. Winning those grants benefits group members by, for example, improving their experimental equipment, human resources, and research materials. Such grants can thus be viewed as a type of public-good prize for group members. A group with more published papers than the others enjoys an advantage in the competition for research grants.

Despite its real-world significance, these characteristics of "one or nothing" and "step-by-step" have not been the focus of previous studies of group contests (e.g., patent races, research joint ventures, political lobbying). The achievement of effort, which is called a group impact function, in the literature on group contests is approximately defined as a continuous function that is straightforward to deal with. Based on the foregoing, this study examines a contest in which the achievement of effort increases step-by-step between groups for a group-specific public-good prize. In the contest, individuals expend effort in their groups to win the prize.

[1]Grant reviewers' decision whether to provide a grant to the group depends not only on their research proposal, but also on the quantity and quality of their published papers. Neufeld, Huber, and Wegner (2013) point out that the journal impact factor and the number of highly cited papers are relevant to the reviewers' judgments on the Starting Grants of the European Research Council. Melin and Danell (2006) show the similar results about a funding program of the Swedish Foundation for Strategic Research.

[2]For instance, the National Science Foundation and National Institutes of Health in the United States provide competitive research grant programs. The KAKENHI (Grants-in-Aid for Scientific Research) program in Japan is another such competitive research grant.

Each individual's small effort is insufficient alone and only becomes beneficial when it is added to the efforts of the other group members. Under this structure, group members can publish a paper only when their research activities reach a quality level suitable for a publication. This process is then repeated. We characterize the Nash equilibria and their existence in step-by-step contests. We find that the existence of the Nash equilibrium always includes an equilibrium at which no members in each group free-ride on the others, but rather expend positive effort. We also show that the effort levels at the Nash equilibria are less than the optimal effort level for each group. Those characteristics at the Nash equilibrium are different from the results on the continuous achievement of effort in previous studies.

Group contests can be classified into two strands of the literature[3]: stochastic competition and deterministic competition (also called all-pay auctions). Under the former, the winner is determined stochastically and competitors' effort increases the winning probability in the contest (Katz, Nitzan, and Rosenberg (1990), Ursprung (1990), Baik (1993), Riaz, Shogren, and Johnson (1995), and Dijkstra (1998)). Under the latter, the competitor who exerts the largest effort becomes the winner (see the original study by Baik, Kim, and Na (2001)). In this study, we focus on a stochastic competition model that has a group impact function with a step-by-step structure. Baik (2008) studies a contest among groups by considering a stochastic competition model in which each individual's effort is beneficial by itself marginally and continuously in each group. The author's main result is that only an individual with a maximum valuation of the public-good prizes exerts effort at the Nash equilibrium, whereas the others free-ride on such effort. In our model of step-by-step group contests, effort adds to the winning probability only when the total effort of a group reaches a certain level. Our results thus differ from those of Baik (2008). At the Nash equilibria in our model, multiple group members expend effort, including an equilibrium at which all members expend effort without any free-riding.

Such stochastic group contests for public-good prizes have been studied expansively. Based on the works of Hirshleifer (1983, 1985), Lee (2012) studies weakest-link contests with group-

[3]Chowdhury and Topolyan (2016) summarize these studies.

specific public-good prizes in which the minimum effort level among group members' effort levels is their group effort or members' effort is a perfect complement. Lee explains that such a structure appears in team competitions of a research contest in which some experts' works are indispensable. In that case, multiple Nash equilibria exist, and a unique coalition-proof Nash equilibrium without free-riding exists. On the contrary, Chowdhury, Lee, and Sheremeta (2013) examine best-shot[4] contests in which the maximum effort level among group members' effort levels is their group effort. They mention that an example of this structure is the case of competing research joint ventures (RJVs) in which a high-quality innovation proposed by one RJV member also benefits the other RJV members. In this case, there are multiple Nash equilibria, and at each equilibrium, only one player in each group at most exerts effort, whereas the others free-ride. Chowdhury and Topolyan (2016) combine both contests. They study the attack-and-defense group contest in which the maximum effort level among group members' effort levels becomes the effort of one group (attacker) and the minimum effort level among group members' effort levels becomes the effort of another group (defender). According to them, such a structure appears in asymmetric patent competitions between parallel multiple R&D teams run by one firm and a sequentially specialized R&D team run by another firm. In this case, multiple Nash equilibria exist at which only one member exerts effort in the attacker group and all members exert effort in the defender group. There is also a unique coalition-proof Nash equilibrium[5] at which the effort level of the defender group is the largest effort of the Nash equilibria. The results of these studies except for the perfect complements case show that only the member with the highest valuation of the public-good prize in each group expends effort, while the others free-ride.

The remainder of the paper is organized as follows. Section 2 presents our model and conditions for the Nash equilibrium. Section 3 presents the existence of the Nash equilibria.

[4]See also the original works of Hirshleifer (1983, 1985).

[5]With regard to the coalition-proof Nash equilibrium, Bernheim, Peleg, and Whinston (1987) originally defined the equilibrium concept. However, the coalition-proof Nash equilibrium of Lee (2012) and Chowdhury and Topolyan (2016) is different from the original concept of Bernheim et al. (1987). Quartieri and Shinohara (2016) provide the details of the difference and redefine the equilibrium concept of Lee (2012) as the group-proof Nash equilibrium.

Section 4 compares the group-optimal effort level with the levels of the Nash equilibria for each group. Section 5 considers the coalition-proof Nash equilibrium. In Section 6, we conclude with a brief discussion of our results. All proofs are in the Appendix.

2 The model

2.1 The groups and those members

We consider contests in which two groups[6] described as G_1 and G_2 compete to win a prize (e.g., a research grant). The prize is a public good within each group called a group-specific public-good prize herein following Baik (2008). G_i, $(i = 1, 2)$ consists of n_i risk-neutral members, $G_i = \{1, 2, 3, \ldots, n_i\}$, where $n_i \geq 2$. They cooperate to win the prize (i.e., they provide effort to their own group). Let v^i_j represent the valuation of the prize of member j in G_i. We assume $0 < v^i_1 \leq v^i_2 \leq \ldots \leq v^i_{n_i}$.

Let $x^i_j \geq 0$ be the effort level provided by member j in G_i and X_i be the total effort level in G_i, that is $X_i = \sum_{j=1}^{n_i} x^i_j$. No players can recover their effort already expended irrespective of whether their group wins the prize. Effort levels are measured in the same unit as the prize values. We assume that the winner of the prize is decided probabilistically and that the winning probability of each group depends on its own and the other group's effort levels. The winning probability of G_i, which is called the contest success function, is described as $p_i(X_1, X_2) = \frac{f(X_i)}{f(X_1)+f(X_2)}$, where $0 \leq p_i \leq 1$ and $p_1 + p_2 = 1$. $f(X_i)$ is called the group impact function following previous studies (Chowdhury, Lee, and Sheremeta (2013), Chowdhury and Topolyan (2016)). Here, we consider a case in which $f(X_i)$ is a step function, defined as $f(X_i) = t$, if $X_i \in [\sum_{k=0}^{t} m_k, \sum_{k=0}^{t+1} m_k)$, $t = 0, 1, 2, \ldots$. In the example of a research grant competition, t means the number of papers published by G_i. The t of G_1 and G_2 is described as α and β, respectively, so that $\alpha, \beta = 0, 1, 2, \ldots$. G_i needs to reach $f(X_i) = 1$, called the "achievement level 1," to acquire a positive winning probability when the other reaches level

[6]For simplicity, we assume two groups. However, even if N groups ($N \geq 2$) are assumed, all the results presented in this paper hold.

1 or higher. This function means that each group needs to raise the effort level, $\sum_{k=0}^{t} m_k$, or more to reach an achievement level t. We define $m_0 = 0$ and $p_i = \frac{1}{2}$ when both $f(X_i)$ are zero[7], for convenience. Then, the winner of the prize is decided by a coin toss, with a probability $\frac{1}{2}$, when both groups do nothing[8]. If it raises less effort than the level $\sum_{k=0}^{t} m_k$, the group cannot reach level t, and some of its effort, $X_i - \sum_{k=0}^{t-1} m_k$, is in vain.

Now, we consider a situation in which the individuals in each group have a lower valuation of the prize than the cost of the next higher achievement level; in other words, we assume $v_j^i < m_k$ for all positive k, all $i \in \{1, 2\}$, and all $j \in G_j$ [9]. Thus, no member has an incentive to expend sufficient effort to reach the next higher achievement level by him/herself. Group members need to collaborate with one another to step up their achievement level. We also assume that the cost for each achievement level is weakly increasing, that is $0 = m_0 < m_1 \leq m_2 \leq \ldots \leq m_k \leq \ldots$. These assumptions mean that, for example, a research group needs to share the tasks of the experiments among members to achieve publication and tries to submit papers to journals one by one. Thus, groups need to expend more effort to reach the same or higher quality results.

Let u_j^i be the expected payoff of member j in G_i. Then, the payoff function for member j in G_i is $u_j^i = p_i(X_1, X_2)v_j^i - x_j^i$. We assume that all members in each group in the contest choose their effort levels x_j^i independently and simultaneously. We assume that all the above is common knowledge among all players. We employ the Nash equilibrium as the equilibrium concept in this game. Thus, the strategy at the equilibrium of this game is an $n_1 + n_2$-tuple vector of each member's effort level in each group, at which each member's effort level is the best response to the other members' effort levels in his/her group and the other group. Here, for any member j in each group, his/her payoff is affected not by particular members' effort levels in each group, but by the total effort level of the others $\sum_{h \neq j} x_h^i$ in his/her group and the total effort level X_{-i} of the other group, namely the other group's achievement level $f(X_{-i})$. Accordingly, even

[7] This is the same definition as the contest success function for group i, p_i, in Baik (2008).

[8] This assumption does not affect the following results intrinsically. We show the same results even if any group obtains nothing when $X_i = 0$ and $f(X_i) = 0$ for all i. See the Introduction of Quartieri and Shinohara (2016).

[9] We can remove this assumption. Instead, we can assume that some members have higher valuations than the costs of achievement, namely $v_j^i \geq m_k$. See also footnote 11.

if each group is on a vector of achievement levels (α, β) at the Nash equilibrium, there may be multiple combinations of each member's effort keeping the vector (α, β). In other words, there are multiple Nash equilibria on a vector of achievement levels (α, β). From the above structure, even if more groups participate in this game, namely N groups, the following results presented in this paper do not change.

2.2 The equilibrium

In this section, we consider the conditions for the Nash equilibrium of the game. Suppose that G_1 can reach the next higher level α from $\alpha - 1$ if member j in G_1 expends efforts $x_j^1 > 0$, given both others' effort levels x_{-j}^1 and the other group's achievement level $f(X_2) = \beta$ of G_2. More accurately, there is some $x_j^1 > 0$ such that $x_j^1 \leq v_j^1$ and $x_j^1 + \sum_{h \neq j} x_h^1 \geq \sum_{k=0}^{\alpha} m_k$. If j expends effort x_j^1, his/her payoff becomes $u_j^1 = \frac{f(X_1)}{f(X_1)+f(X_2)} v_j^1 - x_j^1 = \frac{\alpha}{\alpha+\beta} v_j^1 - x_j^1$. If j does not expend any effort $x_j^1 = 0$, his/her payoff is $u_j^1 = \frac{\alpha-1}{\alpha-1+\beta} v_j^1$, because $\sum_{k=0}^{\alpha} m_k > \sum_{h=1}^{n_i} x_h^1 \geq \sum_{k=0}^{\alpha-1} m_k$; in other words, the achievement level of G_1 remains $\alpha - 1$[10]. When the former is equal to or larger than the latter, that is $\frac{\alpha}{\alpha+\beta} v_j^1 - x_j^1 - \frac{\alpha-1}{\alpha-1+\beta} v_j^1 = \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1 - x_j^1 \geq 0$, j expends effort x_j^1. From this calculation, we obtain the upper limit of j's effort, $x_j^1 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1$. Clearly, $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1 < v_j^i < m_k$. Here, the achievement level of G_1 reaches level α when the total effort of G_1 is equal to or larger than $\sum_{k=0}^{\alpha} m_k$ by j's expense of x_j^1. If $x_j^1 + \sum_{h \neq j} x_h^1 > \sum_{k=0}^{\alpha} m_k$, By keeping level α, j can save his/her effort level by sufficiently small $\epsilon > 0$. Then, he/she can increase his/her payoff by ϵ. As a result, j's optimal strategy is to expend his/her effort for his/her group such that $x_j^1 = \sum_{k=0}^{\alpha} m_k - \sum_{h \neq j} x_h^1$ and $0 \leq x_j^1 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1$. Thus, the best response function of member j of G_1 is

$$x_j^1(x_{-j}^1, X_2) = \begin{cases} \sum_{k=0}^{\alpha} m_k - \sum_{h \neq j} x_h^1 & \text{if } 0 \leq \sum_{k=0}^{\alpha} m_k - \sum_{h \neq j} x_h^1 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1 \\ 0 & \text{otherwise.} \end{cases}$$

A similar argument is applied to each member in G_2. The best response function shows that member j in G_i expends his/her effort without any extra only when his/her effort is beneficial

[10] If j expends any effort less than x_j^1, j has to pay this effort cost on the same achievement level $\alpha - 1$. Thus, in this case, j expends nothing.

for his/her group. The graph of the function is a shape like gradually weakening beats in Figure 1, because $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)}$ is a decreasing function of α, as we show below[11].

Let an n_1+n_2-tuple vector of all players' strategies be $\boldsymbol{x} \equiv (x_1^1, x_2^1, x_3^1, \ldots, x_{n_1}^1; x_1^2, x_2^2, x_3^2, \ldots, x_{n_2}^2)$, which is called a strategy profile hereafter. Then, we have the following lemma.

Lemma 1. *$\boldsymbol{x}^*$ is a Nash equilibrium on the achievement levels (α, β), $\alpha, \beta = 1, 2, 3, \ldots$ if and only if (i) $\sum_{j=1}^{n_i} x_j^{i*} = \sum_{k=0}^{l} m_k$ for all $(i, l) \in \{(1, \alpha), (2, \beta)\}$ and (ii) $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1 \geq x_j^{1*}$ for all $j \in G_1$ and $\frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} v_j^2 \geq x_j^{2*}$ for all $j \in G_2$.*

All proofs are in the Appendix. This lemma leads to an interesting characteristic. At the Nash equilibrium, any member in G_1 expends effort $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1$ less than v_j^1 at most. Those in G_2 do as well. In other words, no member ever expends the same effort level as his/her valuation of the public good because he/she expends only the level of effort that is the difference in the benefits between making effort and not making effort. Lemma 1 also means that if multiple α and β meet both (i) and (ii) in the lemma, there is another Nash equilibrium on another achievement level. In addition, even when there is only a vector of achievement levels (α, β) which meets (i) and (ii) of Lemma 1, if there are multiple strategy profiles $\boldsymbol{x}$ on (α, β), we have multiple Nash equilibria. Figure 2 shows the multiple Nash equilibria in the case of $n_1 = n_2 = 2$. The part at which both the best response curves overlap in the figure is the Nash equilibria on an achievement level (t, s).

In the model of Baik (2008), no matter how little effort each group member expends, it still

[11] As we mentioned in footnote 9, we can remove the assumption of $v_j^k < m_k$. Then, some members with high valuations are allowed to expend sufficient effort by him/herself to cover the cost from the achievement level zero through a positive achievement level $\hat{\alpha} \geq 1$, that is $\sum_{k=0}^{\hat{\alpha}} m_k$. The best response function changes to

$$x_j^1(x_{-j}^1, X_2) = \begin{cases} \sum_{k=0}^{\hat{\alpha}} m_k - \sum_{h \neq j} x_h^1 & \text{if } 0 \leq \sum_{k=0}^{\hat{\alpha}} m_k - \sum_{h \neq j} x_h^1 \leq \frac{\hat{\alpha}}{\hat{\alpha}+\beta} v_j^1 \\ \sum_{k=0}^{\alpha} m_k - \sum_{h \neq j} x_h^1 & \text{if } 0 \leq \sum_{k=0}^{\alpha} m_k - \sum_{h \neq j} x_h^1 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1 \text{ and } \hat{\alpha} \leq \alpha - 1 \\ 0 & \text{otherwise.} \end{cases}$$

The first row on the right-hand side is added in the best response function. In this case, while the analysis becomes complicated, the equilibrium at which only the member with the largest valuation expends effort is just added to the results that we show in the following part, instead of the zero Nash (i.e., no members expend a positive effort).

increases his/her probability of winning the competition marginally because the group impact function, namely the achievement of each member's effort in the group, is continuous. The main result of Baik (2008) is thus that only a member who has the highest valuation of the public-good prize expends effort to his/her group at the Nash equilibrium. Even when each member has his/her own budget of effort, the result is preserved basically. To put it precisely, members expend effort in the order of the high valuation at the Nash equilibrium and those who have lower valuations do not expend effort. On the contrary, the Nash equilibrium in our model does not depend on the order of the valuation. In other words, Lemma 1 states that members who have lower valuations can expend effort to their groups without depending on the order of their valuations at the Nash equilibrium. In particular, condition (i) of Lemma 1 does not specify those members who expend effort to their group at the Nash equilibrium; it only indicates the total effort level that can just cover the cost of the achievement level. At a glance, our model is similar to the case in which each member has his/her own budget in Baik (2008) in respect of the upper limit of the effort level. However, both results differ because of the following facts. In the model of Baik (2008), since the marginal benefit of effort (i.e., a public-good) of the group member with the highest valuation is larger than that of any other member, it covers a larger marginal cost of effort than others' willingness to expend. As a result, others expend nothing and free-ride. In our model, any effort expended is beneficial to each group only when some of the other members expend effort together and their total effort can reach some level. Otherwise, the marginal benefit of anyone's effort is zero. In addition, total effort does not depend on the combination of members.

Each member expends his/her effort by the incremental benefits of stepping up to the next higher level at most; in other words, member j's effort level is $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)}v_j^1$ at most. Thus, the total effort level of G_1 is $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)}\sum_{j=1}^{n_i} v_j^1$ at most for reaching level α. That of G_2 is the same. The next corollary shows the total effort levels of both groups at the Nash equilibrium from Lemma 1.

Corollary 1. $\boldsymbol{x}^*$ *is the Nash equilibrium on achievement levels* (α, β), $\alpha, \beta = 1, 2, 3, \ldots$ *only if* $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)}\sum_{j=1}^{n_1} v_j^1 \geq \sum_{j=1}^{n_1} x_j^{1*} = \sum_{k=0}^{\alpha} m_k$ *and* $\frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)}\sum_{j=1}^{n_2} v_j^2 \geq \sum_{j=1}^{n_2} x_j^{2*} = \sum_{k=0}^{\beta} m_k$.

The conditions of this corollary are not sufficient because (ii) of Lemma 1 is not always met even if the conditions of the corollary hold. The corollary does not exclude the case in which someone expends his/her effort $x_j^1 > \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1$ or $x_j^2 > \frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} v_j^2$.

From this corollary, we see that the achievement level that each group can reach has an upper limit given the opponent's achievement level, once group members are fixed. In addition, considering members' own maximum achievement levels and the opponent's maximum level, there are upper limits of the achievement levels in the Nash equilibria.

Let $\Delta_{\alpha-1}^{\alpha}(\beta) = \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)}$ be the difference in G_1's winning probability from $\alpha - 1$ to α when G_2 is on level β, $\alpha, \beta = 1, 2, 3, \ldots$. Surely, $\Delta_{\alpha-1}^{\alpha}(\beta) > 0$. A decrease in $\Delta_{\alpha-1}^{\alpha}(\beta)$ as α increases means a decline in the effort incentive in G_1 from condition (ii) of Lemma 1. Given β, the gap in the difference from α to $\alpha + 1$ is

$$\Delta_{\alpha}^{\alpha+1}(\beta) - \Delta_{\alpha-1}^{\alpha}(\beta) = -\frac{2\beta}{(\alpha+1+\beta)(\alpha+\beta)(\alpha-1+\beta)} < 0.$$

Thus, $\Delta_{\alpha-1}^{\alpha}(\beta)$ is a strictly decreasing function of α. This shows that the winning probability $p_1 = \frac{\alpha}{\alpha+\beta}$ is a strictly increasing function of α and that it is monotonically and strictly diminishing as α increases[12]. In other words, the higher the level his/her group reaches, the smaller is the effort incentive of each individual. By calculating the additional gap in those differences, we have

$$(\Delta_{\alpha+1}^{\alpha+2}(\beta) - \Delta_{\alpha}^{\alpha+1}(\beta)) - (\Delta_{\alpha}^{\alpha+1}(\beta) - \Delta_{\alpha-1}^{\alpha}(\beta)) = \frac{6\beta}{(\alpha+2+\beta)(\alpha+1+\beta)(\alpha+\beta)(\alpha-1+\beta)} > 0.$$

The decrease in $\Delta_{\alpha}^{\alpha+1}(\beta)$ strictly diminishes as α increases. Noting that $m_0 < m_1 \leq m_2 \leq \ldots$, $\sum_{k=0}^{\alpha} m_k$ is a weakly increasingly increasing function of α. Thus, there is a unique maximum α such that $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} \sum_{j=1}^{n_1} v_j^1 - \sum_{k=0}^{\alpha} m_k \geq 0$ and $\frac{\beta}{(\alpha+\beta)(\alpha+1+\beta)} \sum_{j=1}^{n_1} v_j^1 - \sum_{k=0}^{\alpha+1} m_k < 0$, given β. Naturally, the larger $\sum_{j=1}^{n_1} v_j^1$ in the first term of this inequality, the larger the maximum α tends to be. Concretely, G_1 has a large population, or the group members have high valuations of the prize; then, G_1 tends to have a higher reachable level. A similar argument can be applied in G_2. Given α, the maximum β such that $\frac{\alpha}{(\beta-1+\alpha)(\beta+\alpha)} \sum_{j=1}^{n_2} v_j^2 - \sum_{k=0}^{\beta} m_k \geq 0$ and $\frac{\alpha}{(\beta+\alpha)(\beta+1+\alpha)} \sum_{j=1}^{n_2} v_j^2 - \sum_{k=0}^{\beta+1} m_k < 0$ is obtained.

[12] $\Delta_{\beta-1}^{\beta}(\alpha)$ and p_2 are as well.

Next, given α, the difference from β to $\beta - 1$ of $\Delta^{\alpha}_{\alpha-1}(\beta)$ is

$$\Delta^{\alpha}_{\alpha-1}(\beta) - \Delta^{\alpha}_{\alpha-1}(\beta - 1) = \frac{\alpha - \beta}{(\alpha + 1 + \beta)(\alpha + \beta)(\alpha - 1 + \beta)}. \tag{1}$$

Here, by fixing α, we consider the change in members' effort incentive in G_1 as β increases. When $\beta < \alpha$, (1) is positive. This means that a rise in β (i.e., G_2 reaches the next higher achievement level) provides a larger effort incentive to each member in G_1, who has already been in an advantageous position. Noting that (1) is symmetrical in both groups, the members of G_2 have smaller effort incentives when $\beta < \alpha$. When $\beta = \alpha$ (i.e., both groups have an even chance of winning the prize), (1) is zero. Then, the effort incentive in G_1 is unchanged, even when the opponent's achievement level changes from level $\beta - 1$ to β. When $\beta > \alpha$, (1) is negative. Then, the effort incentive in G_1 drops as β increases. In summary, when an opponent's achievement is at the same level or one lower than its own level, namely β and $\beta - 1$ such that $\alpha = \beta$ (the chance of winning is even), members in both groups can have the maximum effort incentive. Thus, we have the next proposition.

Proposition 1. *Each group member has the maximum effort incentive to reach the next higher achievement level when the opponent group is on the same achievement level.*

These facts lead to an interesting characteristic of the Nash equilibrium. Even if members' effort levels in G_1 and G_2 meet the conditions of Lemma 1 on a high achievement level such that $\alpha = \beta$, condition (ii) of Lemma 1 may not be met on the same achievement level α of G_1 in the range of small β like $\beta = 1$. This means that group members do not have incentives to expend much effort wastefully to reach much higher levels than the opponent. On the contrary, members have effort incentives to be at the same level or one higher than an opponent on a high achievement level. These results indicate that each member in a group tends to expend the effort level at which his/her group can reach the same achievement level as that of the other group because of the competition among groups.

In order for a group to reach a high achievement level at the Nash equilibrium, the opponent also needs to reach a higher level. The next example shows this characteristic.

Example 1. Now, we consider an example where $v_j^i = 8$ for all $j \in G_i$, $i = 1, 2$, $n_1 = n_2 = 40$ and $m_1 = m_2 = 10 < m_3 = m_4 = \ldots = 12$. From condition (ii) of Lemma 1, each member in G_1, given β of G_2, can expend $x_j^1 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1$ for their group to step up from level $\alpha - 1$ to α. Considering condition (i) of Lemma 1, the range of the achievement level α, that is G_1 can reach the Nash equilibrium, is such that (B): $\sum_{j=1}^{40} x_j^1 = \sum_{k=0}^{\alpha} m_k \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} \times 8 \times 40$.

When $(\alpha, \beta) = (3, 1)$, (B) is not met: $\sum_{k=0}^{3} m_k = 32 > \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} = \frac{1}{(3-1+1)(3+1)} 320 = \frac{80}{3}$. In this case, the effort level in G_1 cannot pay for the costs of $\alpha = 3$. When $(\alpha, \beta) = (3, 2)$, (B) is met: $\sum_{k=0}^{3} m_k = 32 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} = \frac{2}{(3-1+2)(3+2)} 320 = 32$. When $(\alpha, \beta) = (3, 3)$, (B) is also met: $\sum_{k=0}^{3} m_k = 32 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} = \frac{3}{(3-1+3)(3+3)} 320 = 32$. Thus, the achievement levels $(\alpha, \beta) = (3, 2), (3, 3)$ can be realized as Nash equilibria at which $\beta = 2$ and 3 provide the same and the maximum effort incentives to members in G_1. In fact, for $\beta > 3$, the difference in the winning probability of G_1 is monotonically decreasing: $\Delta_{\alpha-1}^{\alpha}(\beta) = \Delta_3^4(1) = \frac{1}{12} < \Delta_3^4(2) = \Delta_3^4(3) = \frac{1}{10} > \Delta_3^4(4) = \frac{2}{21} > \Delta_3^4(5) = \frac{5}{56} > \ldots$. Furthermore, members in G_1 cannot pay for the costs $\sum_{k=0}^{4} m_k = 44$, $\sum_{k=0}^{5} m_k = 56, \ldots$ in $\beta > 3$. The same argument is applied in G_2 because of symmetry.

If α is small, those α are reachable as Nash equilibria even for small β. When $(\alpha, \beta) = (2, 1)$, (B) is met: $\sum_{k=0}^{2} m_k = 20 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} = \frac{1}{(2-1+1)(2+1)} 320 = \frac{160}{3}$. When $(\alpha, \beta) = (2, 2)$, (B) is also met: $\sum_{k=0}^{2} m_k = 20 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} = \frac{2}{(2-1+2)(2+2)} 320 = \frac{160}{3}$.

On the contrary, G_1 cannot always reach a higher α even when β is the same level as α. When $(\alpha, \beta) = (4, 4)$, (B) is not met: $\sum_{k=0}^{4} m_k = 44 > \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} = \frac{4}{(4-1+4)(4+4)} 320 = \frac{160}{7}$. Thus, when the population of groups is $n_1 = n_2 = 40$, who has the evaluation of $v_j^i = 8$, the maximum achievement level that both groups can reach is $(\alpha, \beta) = (3, 3)$. □

3 Existence of a Nash equilibrium

In this section, we show the existence of a Nash equilibrium. We can obtain the condition for the existence of a Nash equilibrium from Lemma 1 and Corollary 1.

Lemma 2. *There is at least one Nash equilibrium on positive achievement levels* (α, β), $\alpha, \beta = 1, 2, 3, \ldots$ *if and only if* $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} \sum_{j=1}^{n_1} v_j^1 \geq \sum_{k=0}^{\alpha} m_k$ *and* $\frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} \sum_{j=1}^{n_2} v_j^2 \geq \sum_{k=0}^{\beta} m_k$.

This lemma mentions that there is at least one Nash equilibrium at which each group reaches a positive achievement level when the difference in members' valuations of the public-good prize from an achievement level to the next higher level is as much as or more than the costs of the achievement levels. Hereafter, this is called a positive Nash.

If we can allocate positive effort levels to all group members such that the conditions of Lemma 1 hold, we obtain a Nash equilibrium at which all players expend positive effort levels without any free-riders. For example, see the rule of allocating costs in proportion to each member's valuation share in each group: $x_j^i = \frac{v_j^i}{\sum_{j=1}^{n_i} v_j^i} \sum_{k=0}^{t} m_k$ for all $j \in G_i$, $i = 1, 2$ and $t = \alpha, \beta$. This is called "Expense Rule A" hereafter. Then, we have the next proposition from Lemma 1 and Lemma 2.

Proposition 2. *There is always at least one Nash equilibrium at which each group member expends a positive effort level on each achievement level* (α, β) *such that the conditions of Lemma 2 hold.*

From Lemma 2 and Proposition 2, the next theorem is obtained.

Theorem 1. *There is at least one Nash equilibrium at which all group members expend positive effort if and only if* $\frac{1}{2} \sum_{j=1}^{n_i} v_j^i \geq m_1$.

Theorem 1 means that the existence of at least one Nash equilibrium, which includes each member who expends a positive effort level, is equivalent to each group having half of the valuation of the group members that have as much as or more than the cost of the level (i.e., the effort for the first achievement). Otherwise, no member expends any effort.

In Baik (2008), Chowdhury et al. (2013), and Chowdhury and Topolyan (2016), only a group member expends effort and others free-ride at the Nash equilibrium. On the contrary, in our model, there is at least one Nash equilibrium at which all members in all groups expend positive effort levels on each achievement level such that the conditions of Lemma 2 hold. This result comes from the structure, namely that each group member's effort is effective discretely only when those efforts are gathered. In addition, in step-by-step group contests, expending effort in each group has the same structure as a coordination game. Accordingly, it is possible to achieve strategy profile $\boldsymbol{x}$ as a Nash equilibrium at which each member is allocated a positive effort in a group with Expense Rule A. Indeed, the results of previous studies, namely that only one member expends effort and the others free-ride, are not always consistent with the real-world case of research grant competition, when many group members make effort to achieve publications. The results of our model thus explain the case well when all group members expend effort.

However, in any v_j^i and m_k, unrelated to the conditions of Lemma 2, there is a Nash equilibrium such that $x_j^i = 0$ for all i and j; as a result, $f(X_1) = f(X_2) = 0$. In Figure 2, this is illustrated for the case of $n_1 = n_2 = 2$. The proof is similar to that of Lemma 1. Suppose, given the others' effort levels $x_{-j}^i = 0$, that any member j in G_i deviates from $x_j^i = 0$ to some $x_j^i > 0$. If the achievement level of G_i does not reach $f(X_i) = 1$ following this deviation, j does not deviate, because that brings him/her a decrease in his/her payoff by the cost of effort x_j^i. If G_i reaches the achievement level 1 because of j's effort, he/she has an incentive to expend positive effort $x_j^i > 0$ to his/her group because of $v_j^i - x_j^i \geq \frac{1}{2}v_j^i - 0$. However, this is impossible because his/her rewarding effort cannot cover the cost: $m_1 > v_j^i > \frac{1}{2}v_j^i \geq x_j^i$. Thus, $\boldsymbol{x} = \boldsymbol{0}$ is always a Nash equilibrium. Hereafter, this is called the zero Nash[13].

Note the assumption that a public-good prize is allocated to either G_1 or G_2 when $\boldsymbol{x} = \boldsymbol{0}$. However, we may not obtain anything without members' effort in the real world, even if our opponents do nothing. In our model, we thus obtain the same results even when we assume that no group wins anything when $f(X_i) = 0$ for all i. Thus, our model can explain both cases. In other words, those cases have the same structure of the game intrinsically.

In addition, note the assumption that the value of the public-good prize does not increase even if some group members expend effort. Under both assumptions, the zero Nash is more efficient than any positive Nash for all groups[14]. These assumptions represent the case in which members' effort does not add any value. In the example of grant competition, the grant budget is usually decided by the government or founders in advance. Grants are then allocated to the

[13]Even if we assume that no group obtains anything when $X_i = 0$ for all i, j does not deviate from $x_j^i = 0$ because $m_1 > v_j^i > x_j^i > 0$.

[14]However, much research effort brings about many social benefits. This factor is not included in the model.

winner within the set budget. In this study, the assumption that the value of the prize does not increase can be viewed as the budget being determined in advance[15].

Under the condition of Lemma 2, we have the following two corollaries with regard to the range of reachable achievement levels at Nash equilibria.

Corollary 2. *If there is at least one Nash equilibrium on an achievement level $\alpha = \beta = t \geq 2$, there is at least one Nash equilibrium on an achievement level of $\alpha = \beta = t - 1$.*

By applying this corollary repeatedly, we obtain Nash equilibria on the positive achievement levels from $\alpha = \beta = t$ to $\alpha = \beta = 1$. A similar logic to Corollary 2 can be applied to $(\alpha, \beta) = (t, t-1)$ and $(t-1, t)$.

Corollary 3. *If there is at least one Nash equilibrium on an achievement level $\alpha = \beta = t \geq 2$, there is at least one Nash equilibrium on $(\alpha, \beta) = (t, t-1)$ and $(t-1, t)$, respectively.*

In summary, Corollary 2 and Corollary 3 state that once there is at least one Nash equilibrium on achievement levels $(\alpha, \beta) = (t, t)$, $t \geq 2$, there are always Nash equilibria on all (α, β), $(\alpha - 1, \beta)$ and $(\alpha, \beta - 1)$ such that $0 \leq \alpha = \beta \leq t$. Note that as we showed in Example 1, there are not always Nash equilibria on the other achievement levels of $(t, t-r)$ or $(t-r, t)$, $2 \leq r \leq t$. This result means that when the other group expends a large effort level, the first group also expends a large effort level. In other words, each member is given a larger incentive to expend effort by the group contest than simply supplying the discrete public good in a group without any group contests. Accordingly, we can say that the incentive is brought about by the competitive environment among groups.

4 Comparison

Is the total effort level optimal for the group? At a Nash equilibrium at which all group members expend positive efforts, the total level may not be the best for the group. In this section, we compare the Nash equilibria in the previous section with those in a benchmark case

[15] We may also consider the case in which the budget rises. In this case, since each member's valuation increases as more effort is expended, the analysis is more complex.

in which each group makes a decision as one player. We then show the difference between total effort levels under group decisions and individual decisions. Here, the two groups decide their achievement levels to maximize the sum of all members' payoffs in their group:

$$U^i(f(X_1), f(X_2)) \equiv \sum_{j=1}^{n_i} u_j^i = \frac{f(X_i)}{f(X_1)+f(X_2)} \sum_{j=1}^{n_i} v_j^i - \sum_{k=1}^{t} m_k.$$

When these two groups decide their total effort levels X_i to maximize $U^i(f(X_1), f(X_2))$ (i.e., groups decide their achievement levels), how much total effort do they expend? To maximize their payoff, we calculate the difference in $U_1(f(X_1), f(X_2))$ from $\alpha - 1$ to α given the other group's achievement level. This difference is defined as $\bar{\Delta}_{\alpha-1}^{\alpha}(\beta) = U^1(\alpha, \beta) - U^1(\alpha-1, \beta)$, which represents the increment in G_1's payoff by stepping up its achievement level from $\alpha - 1$ to α. We have

$$\begin{aligned}
\bar{\Delta}_{\alpha-1}^{\alpha}(\beta) &= \left(\frac{\alpha}{\alpha+\beta}\sum_{j=1}^{n_1} v_j^1 - \sum_{k=1}^{\alpha} m_k\right) - \left(\frac{\alpha-1}{\alpha-1+\beta}\sum_{j=1}^{n_1} v_j^1 - \sum_{k=1}^{\alpha-1} m_k\right) \\
&= \frac{\beta}{(\alpha+\beta)(\alpha-1+\beta)}\sum_{j=1}^{n_1} v_j^1 - m_\alpha \\
&= \Delta_{\alpha-1}^{\alpha}(\beta)\sum_{j=1}^{n_1} v_j^1 - m_\alpha. \qquad (2)
\end{aligned}$$

(2) seems like the sum of the net marginal utility of α. Suppose that the condition of Theorem 1 holds; in other words, there is at leas one Nash equilibrium on a positive achievement level $\alpha, \beta \geq 1$. Then, given β, we have $\frac{1}{1+\beta}\sum_{j=1}^{n_1} v_j^1 - m_1 \geq 0$ of the condition of Lemma 2 at $\alpha = 1$, which is the same formula as (2) at $\alpha = 1$. Note that $\Delta_{\alpha-1}^{\alpha}(\beta)$ is a strictly decreasing function of α from the calculation in the previous section and that $m_\alpha \leq m_{\alpha+1}$ for any α. Clearly,

$$\bar{\Delta}_{\alpha}^{\alpha+1}(\beta) - \bar{\Delta}_{\alpha-1}^{\alpha}(\beta) = -\frac{2\beta}{(\alpha-1+\beta)(\alpha+\beta)(\alpha+1+\beta)} - (m_{\alpha+1} - m_\alpha) < 0.$$

From this inequality, $\bar{\Delta}_{\alpha-1}^{\alpha}(\beta)$ is a monotonically decreasing function of α. If $\bar{\Delta}_0^1(\beta) < 0$, G_1 does not expend any effort. However, that case does not occur under Theorem 1. Since $\bar{\Delta}_0^1(\beta) \geq 0$, G_1 expends some effort to reach a positive achievement level. Thus, given β, we have the maximum payoff at $\alpha \geq 1$ such that $\bar{\Delta}_{\alpha-1}^{\alpha}(\beta) \geq 0$ and $\bar{\Delta}_{\alpha}^{\alpha+1}(\beta) < 0$. In other words, such an α is the best response function to β in the case of a group decision. Then, we have the following lemma.

Lemma 3. *Given the achievement level of the other group, the achievement level that one group can reach when each individual maximizes his/her payoff is the same as or lower than the achievement level when each group maximizes its total payoff.*

When (2) is nonnegative, that condition is similar to Samuelson's condition. On the contrary, the condition of individual decision $\Delta_{\alpha-1}^{\alpha}(\beta) \sum_{j=1}^{n_1} v_j^1 - \sum_k^{\alpha} m_k \geq 0$ is similar to that of a Nash equilibrium in supplying a public-good. When the achievement levels of the effort in each group rise step-by-step, the total effort level in the individual decision case is as much as or less than the group optimal level for each group. This result comes from the characteristic that each group member expends only effort at most by the incremental expected benefit when his/her group reaches the next higher achievement level in the best response to others' effort levels.

The strategy profile of a Nash equilibrium in the group decision case on (α, β) needs to meet $\bar{\Delta}_{\alpha-1}^{\alpha}(\beta) \geq 0$, $\bar{\Delta}_{\alpha}^{\alpha+1}(\beta) < 0$, $\bar{\Delta}_{\beta-1}^{\beta}(\alpha) \geq 0$, and $\bar{\Delta}_{\beta}^{\beta+1}(\alpha) < 0$. In the next example, we compare the reachable achievement levels in the group decision with those in each individual decision.

Example 2. We reconsider Example 1: $v_j^i = 8$ for all $j \in G_i$, $i = 1, 2$, $n_1 = n_2 = 40$ and $m_1 = m_2 = 10 < m_3 = m_4 = \ldots = 12$. Then, (2) is $\bar{\Delta}_{\alpha-1}^{\alpha}(\beta) = \frac{\beta}{(\alpha+\beta)(\alpha-1+\beta)} \times 8 \times 40 - m_k$. First, we check whether $(\alpha, \beta) = (3, 3)$, which is the upper limit of the reachable achievement level in the case of each individual decision, is reachable in the group decision case. When the achievement level is $(3, 3)$, $\bar{\Delta}_2^3(3) = \frac{3}{(3+3)(3-1+3)} \times 8 \times 40 - 12 = 20 > 0$. Thus, $(3, 3)$ is also a reachable achievement level in the group decision case. However, $(3, 3)$ does not bring about the maximum payoff to each group given the other because, given $\beta = 3$, G_1 can increase its payoff by $\frac{3}{(4+3)(4-1+3)} \times 8 \times 40 - 12 = \frac{76}{7}$ by deviating from $\alpha = 3$ to 4.

Next, we check the Nash equilibrium in the group decision case. When $(7, 7)$, $\bar{\Delta}_6^7(7) = \frac{7}{(7+7)(7-1+7)} \times 8 \times 40 - 12 = \frac{4}{13} > 0$. This means that G_1's payoff increases by $\frac{4}{13}$ by stepping up from $\alpha = 6$ to 7, given $\beta = 7$. If G_1 steps up from $\alpha = 7$ to 8 given $\beta = 7$, $\bar{\Delta}_7^8(7) = \frac{7}{(8+7)(8-1+7)} \times 8 \times 40 - 12 = -\frac{4}{3} < 0$. This stepping up brings G_1 a decrease in its payoff by $\frac{4}{3}$. Noting that (2) is strictly and monotonically decreasing as α increases, $\alpha = 7$, given $\beta = 7$, is the unique maximum point. The same argument applies to G_2 owing to symmetry. Thus, (7, 7) is a Nash equilibrium in the group decision case.

In the case of each individual decision, the highest reachable achievement level of each group is (3, 3) at the Nash equilibria. In the case of each group decision, the highest reachable achievement is (7, 7) at the Nash equilibrium. In Lemma 3, we have the result that given the other group's achievement level, the reachable achievement level in the group decision case is as high as or higher than that in each individual decision generally[16]. This example shows that the level in the group decision case can be a few times higher than that in the individual decision case. □

5 Consideration of the coalition-proof Nash equilibrium

In step-by-step group contests, there are multiple Nash equilibria, as we showed in the previous sections. In our model, as well as in Lee (2012) and Chowdhury et al. (2013), we can thus employ the coalition-proof Nash equilibrium. When we carefully examine the coalition-proof Nash equilibrium of Lee (2012) and Chowdhury et al. (2013), they use this equilibrium concept without any strict definition. The coalition-proof Nash equilibrium defined by Bernheim et al. (1987) is generally used in game theory. However, the coalition-proof Nash equilibrium of Lee (2012) and Chowdhury et al. (2013) does not share the same definition as that of Bernheim et al. (1987). Quartieri and Shinohara (2016) point out this problem, and provide the detailed considerations of the differences between Lee (2012) and Bernheim et al. (1987), strictly redefining the coalition-proof Nash equilibrium of Lee (2012) as the group-proof Nash equilibrium[17]. Briefly, the coalition-proof Nash equilibrium of Lee (2013) and Chowdhury et al. (2013) considers the possibility of communication among only group members rather than with the members of other groups, while that of Bernheim et al. (1987) considers the possibility of communication among all players. Since it seems implausible in group contests that group members would communicate with those in rival groups and agree to deviate to

[16]If (2) is nonnegative only in $(\alpha, \beta) = (1, 1)$, both cases are the same level at a Nash equilibrium.

[17]See Bernheim et al. (1987) and Quartieri and Shinohara (2016) for the strict definitions of the coalition-proof Nash equilibrium and group-proof Nash equilibrium, respectively.

another strategy, the group-proof Nash equilibrium is a more appropriate refinement for use in the contests setting, as Quartieri and Shinohara (2016) mention.

The group-proof Nash equilibrium does not always bring about uniqueness in our model in contrast to the models of Lee (2012) and Chowdhury et al. (2013). We reconsider Example 1. The achievement levels of $(\alpha, \beta) = (2, 2)$ and $(3, 3)$ are at the Nash equilibria, as we checked in the example. For instance, $\boldsymbol{x}^* = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ on $(\alpha, \beta) = (2, 2)$ and $\boldsymbol{x}^{**} = (\frac{4}{5}, \frac{4}{5}, \ldots, \frac{4}{5}; \frac{4}{5}, \frac{4}{5}, \ldots, \frac{4}{5})$ on $(3, 3)$ are Nash equilibria strategies because condition (B) of Example 1 is met at $\boldsymbol{x}^*$ and at $\boldsymbol{x}^{**}$. In addition to $\boldsymbol{x}^{**}$, $\boldsymbol{x}^*$ is also a group-proof Nash equilibrium. Even if fewer than 40 members communicate and agree to deviate from $\frac{1}{2}$ to any other strategy recursively, given the others' strategies, this does not benefit any member in the coalition. In fact, suppose that at $\boldsymbol{x}^*$, 39 members without only member j communicate and agree to change $x^{i*}_{-j} = \frac{1}{2}$ to $\frac{4}{5}$, which meets (B) of Example 1 and which is only a strategy with the possibility of being more beneficial for each member in the coalition when their group reaches the achievement level 3 from 2. However, the group cannot reach the achievement level 3 because $\sum_{j=1}^{n_i} x^i_j = \frac{4}{5} \times 39 + \frac{1}{2} = 31.7 < \sum_{k=0}^{3} m_k = 32$. In addition, any strategies less than $\frac{1}{2}$ by any member coalitions bring about less benefit. Thus, we cannot always obtain a unique group-proof Nash equilibrium under the discrete contest success function.

Indeed, all Nash equilibria on low achievement levels are eliminated by employing the group-proof Nash equilibrium because three members in a group communicate and agree to expend effort level $\frac{10}{3}$ at the zero Nash, given the others' zero effort, which is a Nash equilibrium strategy on $(1, 0)$, $(0, 1)$, and $(1, 1)$. In addition, any Nash equilibria on $(1, 1)$ are excluded. When 15 members at most in a group communicate and agree to deviate from any Nash equilibria strategy on $(1, 1)$ to $\frac{4}{3}$, the group reaches achievement level 2 and all group members then benefit more. Thus, any Nash equilibria on low achievement levels can be eliminated by the group-proof Nash equilibrium.

6 Conclusion

In this study, we focused on step-by-step group contests with group-specific public-good prizes, such as research grant competition. In particular, we investigated how the behaviors of free-riders in previous studies change when group members' effort is effective only when it is summed and reaches some achievement level. A discrete step function was introduced as a group impact function in our model in contrast to the standard continuous, maximum, or minimum functions adopted in previous studies. We showed the necessary and sufficient conditions for the Nash equilibrium and those for its existence. We also showed that there is always at least one Nash equilibrium at which each group member expends a positive effort level when a positive achievement level is reachable. These results are different from those of previous studies in which some group members do not expend any effort at the Nash equilibria. The results explain that in the real world, all members often expend some positive effort in research groups at universities owing to the characteristics of step-by-step group competition. However, each member expends at most only an incremental amount of his/her benefit by moving up to the next higher achievement level. This effort is less than or the same as, at most, the effort when a group maximizes its own benefits. In addition, by surveying the concepts of the coalition-proof Nash equilibrium and the group-proof Nash equilibrium, we showed that the group-proof Nash equilibrium does not always bring about uniqueness despite refining the Nash equilibria on low achievement levels.

In future research, we may consider the case in which each individual valuation of group-specific public-good prizes increases as members' effort levels increase. In grant competition, the effort that group members expend does not increase the provision of research grants. In the real world, effort in groups adds some value. However, the characterization of the Nash equilibrium in this case would be complicated because the results depend on how the effort adds value. This issue needs to be studied in future research.

Appendix: Proofs

Proof of Lemma 1. (1) Necessity. Suppose that $\boldsymbol{x}^*$ is a Nash equilibrium at which G_1 and G_2 are on achievement levels α and β, respectively and $\boldsymbol{x}^*$ does not meet (i) or (ii), that is for some $(i, l) \in \{(1, \alpha), (2, \beta)\}$, (i') $\sum_{j=1}^{n_i} x_j^{i*} > \sum_{k=0}^{l} m_k$ or (i") $\sum_{j=1}^{n_i} x_j^{i*} < \sum_{k=0}^{l} m_k$, or (ii') $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1 < x_j^{1*}$ for some $j \in G_1$, or $\frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} v_j^2 < x_j^{2*}$ for some $j \in G_2$.

(i') contradicts the best response of j because decreasing j's effort that is sufficiently small $\epsilon > 0$ increases his/her payoff by ϵ. (i") means that neither G_1 nor G_2 is on α or β, or both G_1 and G_2 are not on α or β. Thus, these contradict that $\boldsymbol{x}^*$ is a Nash equilibrium on α and β. Noting that the upper limit of each member's effort is $x_j^1 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1$ for each $j \in G_1$ and $x_j^2 \leq \frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} v_j^2$ for each $j \in G_2$ in the best responses, (ii') contradicts these limits; in other words, some small $x_j^i < x_j^{i*}$ brings a better payoff to j. Thus, (i) and (ii) are necessary conditions.

(2) Sufficiency. Suppose that $\boldsymbol{x}^*$, which meets both (i) and (ii), is not a Nash equilibrium. Given the others' effort, one of the members of G_1, say $j \in G_1$, is made better off by deviating from x_j^{1*} to x_j^1, (A): $P_1(X_1^*, X_2^*) v_j^1 - x_j^{1*} < P_1(X_1, X_2^*) v_j^1 - x_j^1$.

First, suppose that $x_j^1 > x_j^{1*}$. G_1 needs to reach at least one higher achievement level $\alpha + 1$ for the increase in j's effort to obtain $P_1(X_1, X_2^*) > P_1(X_1^*, X_2^*)$. The condition under which j's payoff increases by changing his/her effort is

$$\begin{aligned} (\tfrac{\alpha+1}{\alpha+1+\beta} v_j^1 - x_j^1) - (\tfrac{\alpha}{\alpha+\beta} v_j^1 - x_j^{1*}) &> 0 \\ \iff \qquad \tfrac{\beta}{(\alpha+1+\beta)(\alpha+\beta)} v_j^1 &> x_j^1 - x_j^{1*}. \end{aligned}$$

Noting that $v_j^1 < m_k$ for all positive k and that the coefficient of v_j^1 on the left-hand side is less than one, we have $m_k > v_j^1 > \frac{\beta}{(\alpha+1+\beta)(\alpha+\beta)} v_j^1 > x_j^1 - x_j^{1*} > 0$. This means that G_1 cannot reach one higher achievement level $\alpha + 1$ by only j's increase in effort. This contradicts (A).

Second, suppose that $x_j^1 < x_j^{1*}$. Then, the achievement level of G_1 drops from α to $\alpha - 1$ once j decreases his/her efforts from (i). The condition under which j benefits more by decreasing his/her effort is

$$\begin{aligned} (\tfrac{\alpha-1}{\alpha-1+\beta} - x_j^1) - (\tfrac{\alpha}{\alpha+\beta} v_j^1 - x_j^{1*}) &> 0 \\ \iff \qquad \tfrac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1 &< x_j^{1*} - x_j^1 \leq x_j^{1*}. \end{aligned}$$

The last inequality contradicts (ii).

A similar argument is applied when the deviant is a member of G_2. Thus, (i) and (ii) are also sufficient conditions. $\square$

Proof of Corollary 1. By using (i) of Lemma 1, the sum of condition (ii) of Lemma 1 of all the members in G_i is the corollary's condition. $\square$

Proof of Lemma 2. (1) Necessity. Suppose that the condition is not met when there is at least one Nash equilibrium on some (α, β). Then, there is at least one pair of (α, β) meets $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} \sum_{j=1}^{n_1} v_j^1 < \sum_{k=0}^{\alpha} m_k$ or $\frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} \sum_{j=1}^{n_2} v_j^2 < \sum_{k=0}^{\beta} m_k$ at the Nash equilibrium. These inequalities contradict Corollary 1. Thus, the condition is a necessity.

(2) Sufficiency. When the condition is met, there is at least one strategy profile $\boldsymbol{x}$ at which we can allocate all members to expend effort levels for their own groups such that $x_j^1 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1$, $x_j^2 \leq \frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} v_j^2$, $\sum_{j=1}^{n_1} x_j^1 = \sum_{k=1}^{\alpha} m_k$ and $\sum_{j=1}^{n_2} x_j^2 = \sum_{k=1}^{\beta} m_k$. For example, we allocate some members from member 1 in ascending order in each group their full effort level, $x_j^1 = \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1$ and $x_j^2 = \frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} v_j^2$, allocate only last one member $x_j^1 \leq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1$ and $x_j^2 \leq \frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} v_j^2$, and allocate the others $x_j^i = 0$, such that $\sum_{j=1}^{n_1} x_j^1 = \sum_{k=0}^{\alpha} m_k$ and $\sum_{j=1}^{n_2} x_j^2 = \sum_{k=0}^{\beta} m_k$. From Lemma 1, this $\boldsymbol{x}$ is a Nash equilibrium. Thus, the condition is sufficient. $\square$

Proof of Proposition 2. From Lemma 2, we can allocate each member in each group at least one positive effort level, $x_j^{i*} > 0$, for all i and j, by using Expense Rule A, such that the conditions of Lemma 1: $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} v_j^1 \geq x_j^{1*}$, $\frac{\alpha}{(\alpha-1+\beta)(\alpha+\beta)} v_j^2 \geq x_j^{2*}$, $\sum_{j=1}^{n_1} x_j^{1*} = \sum_{k=0}^{\alpha} m_k$ and $\sum_{j=1}^{n_2} x_j^{2*} = \sum_{k=0}^{\beta} m_k$. In fact, we multiply both sides of the conditions of Lemma 2 by $\frac{v_j^i}{\sum_{j=1}^{n_i} v_j^i}$. Then, we have the conditions of Lemma 1. Thus, this proposition holds.

Such $\boldsymbol{x}^*$ is a strategy profile at a Nash equilibrium. Suppose that anyone, say member h, in G_1 deviates his/her effort level from $x_h^{1*} > 0$ to $x_h^1 < x_h^{1*}$, given the other members' effort level in each group. Then, since $\sum_{k=0}^{\alpha} m_k = \sum_{j=1}^{n_1} x_j^{1*} > \sum_{j \neq h}^{n_1} x_j^{1*} + x_h^1$, the achievement level of G_1 declines from α to $\alpha-1$. The change in h's payoff is $\left(\frac{\alpha-1}{\alpha-1+\beta}v_h^1 - x_h^1\right) - \left(\frac{\alpha}{\alpha+\beta}v_h^1 - x_h^{1*}\right) = -\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)}v_h^1 + x_h^{1*} - x_h^1 < -\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)}v_h^1 + x_h^{1*} \leq 0$ from the condition of Lemma 1. h's payoff decreases strictly following this deviation. The same logic applies to anyone in G_2. Thus, $\boldsymbol{x}^*$ induces a Nash equilibrium. $\square$

Proof of Theorem 1. (1) Necessity. Suppose that there is at least one Nash equilibrium on the positive achievement levels (α, β), $\alpha \geq 1$, and $\beta \geq 1$. From Proposition 2, there is also at least one Nash equilibrium at which each member in each group expends a positive effort level on the achievement levels. In addition, we have $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} \sum_{j=1}^{n_1} v_j^1 \geq \sum_{k=0}^{\alpha} m_k$ from Lemma 2. Noting the coefficient of the summation of the valuation on the left-hand side, we have $\frac{1}{2} \geq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)}$ for all $\alpha \geq 1$ and $\beta \geq 1$ because of $\frac{1}{2} - \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} = \frac{\beta(\beta-1)+2(\alpha-1)\beta+\alpha(\alpha-1)}{2(\alpha-1+\beta)(\alpha+\beta)} \geq 0$. Accordingly,

$$\frac{1}{2}\sum_{j=1}^{n_1} v_j^1 \geq \frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} \sum_{j=1}^{n_1} v_j^1 \geq \sum_{k=0}^{\alpha} m_k \geq m_1.$$

The same calculation applies to G_2. Hence, $\frac{1}{2}\sum_{j=1}^{n_i} v_j^i \geq m_1$ is a necessary condition.

(2) Sufficiency. Suppose that the condition is met. Since the condition of Lemma 2 is $\frac{1}{2}\sum_{j=1}^{n_1} v_j^1 \geq m_1$ at $(\alpha, \beta) = (1, 1)$, there is at least one Nash equilibrium at which each member in each group expends a positive effort level on the achievement levels $(1, 1)$, according to Lemma 2 and Proposition 2. $\square$

Proof of Corollary 2. Suppose that Lemma 2 holds on $\alpha = \beta = t$, $t \geq 2$. Noting that $\Delta_{t-1}^t(t) = \frac{1}{2(2t-1))}$ on $\alpha = \beta = t$, we compare the benefit and cost on t with those at $t-1$. For all $t \geq 2$,

$$\frac{1}{2(2(t-1)-1)}\sum_{j=1}^{n_i} v_j^i > \frac{1}{2(2t-1)}\sum_{j=1}^{n_i} v_j^i \geq \sum_{k=0}^{t} m_k > \sum_{k=0}^{t-1} m_k. \tag{3}$$

This inequality indicates that Lemma 2 also holds at $\alpha = \beta = t-1$ when that is at $t \geq 2$. Thus, there is also at least one Nash equilibrium at $t-1$ if there is at least one Nash equilibrium at t. $\square$

Proof of Corollary 3. Suppose that there is at least one Nash equilibrium on an achievement level $\alpha = \beta = t \geq 2$. From (1), $\Delta_{t-1}^{t}(t) = \frac{1}{2(2t-1))} = \Delta_{t-1}^{t}(t-1)$ on $\alpha = \beta = t$. Then, $\frac{1}{2(2t-1)} \sum_{j=1}^{n_i} v_j^i \geq \sum_{k=0}^{t} m_k$ holds in one group G_i on the other group's achievement levels t and $t-1$. When the achievement level in G_i is t, $\frac{1}{2(2(t-1)-1)} \sum_{j=1}^{n_{-i}} v_j^{-i} > \sum_{k=0}^{t-1} m_k$ holds in the other group G_{-i} from (3) in the proof of Corollary 2. From Lemma 2, there is at least one Nash equilibrium on $(\alpha, \beta) = (t, t-1)$. The same logic applies in the case of the achievement level $(t-1, t)$. □

Proof of Lemma 3. α for maximizing the group's payoff has to meet $\bar{\Delta}_{\alpha-1}^{\alpha}(\beta) \geq 0$ and $\bar{\Delta}_{\alpha}^{\alpha+1}(\beta) < 0$. On the contrary, from Corollary 1, the achievement level of G_1 in the case in which each individual maximizes his/her payoffs meets $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} \sum_{j=1}^{n_1} v_j^1 - \sum_{k=0}^{\alpha} m_k \geq 0$. Note that in the previous section, $\Delta_{\alpha-1}^{\alpha}(\beta)$ is a strictly decreasing function of α and $\sum_{k=1}^{\alpha} m_k \geq m_\alpha$ for all $\alpha \geq 1$. Comparing $\frac{\beta}{(\alpha-1+\beta)(\alpha+\beta)} \sum_{j=1}^{n_1} v_j^1 - \sum_{k=0}^{\alpha} m_k \geq 0$ with $\frac{\beta}{(\alpha+\beta)(\alpha-1+\beta)} \sum_{j=1}^{n_1} v_j^1 - m_\alpha \geq 0$, α in the case of maximizing each individual's payoff is the same level as or less than that in the case of each group maximizing its payoff for any $\beta \geq 1$. A similar argument applies in G_2. □

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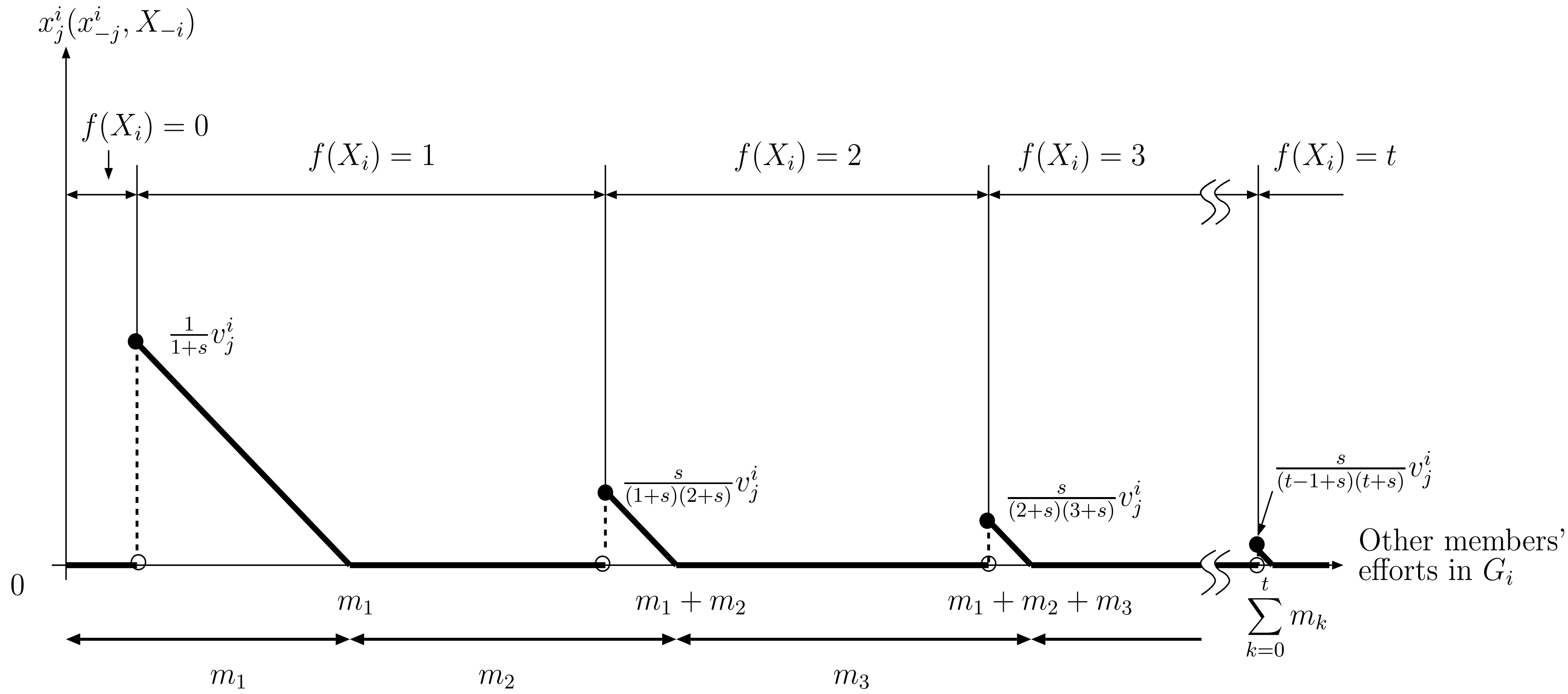

Figure 1: Best response function of member j in G_i at $f(X_{-i}) = s$

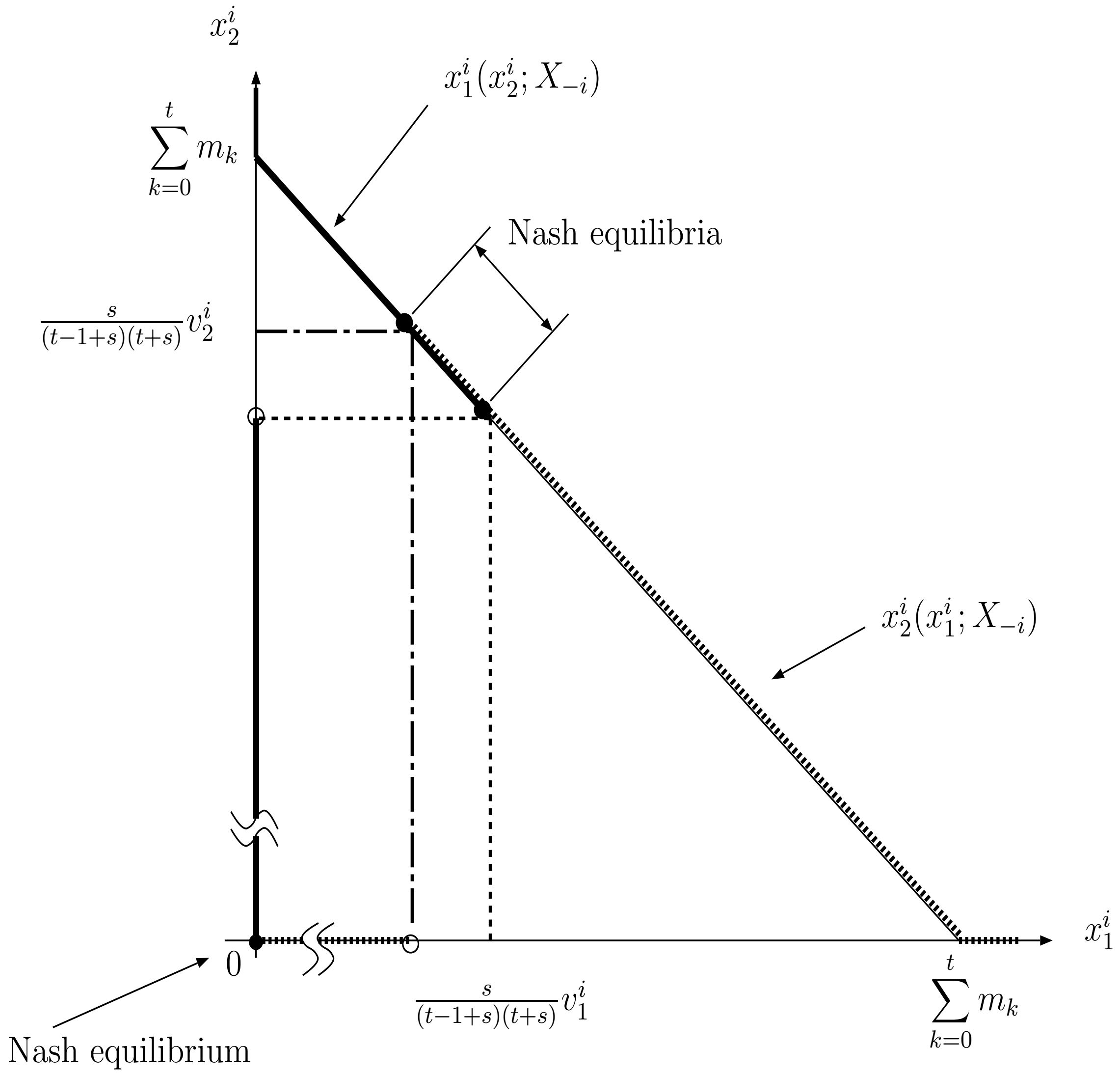

Figure 2: Nash equilibria in each group on $(f(X_i), f(X_{-i})) = (t, s)$ in $n_1 = n_2 = 2$